RECAPTURING A HOLOMORPHIC FUNCTION ON AN ANNULUS FROM ITS MEAN BOUNDARY VALUES

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Abstract. Let $D$ be an annulus in the complex plane with closure $\overline{D}$ and boundary $\partial D$. We prove that a function $f$, holomorphic in $D$ with $C^{1+\epsilon}(\partial D)$ boundary data for some $\epsilon > 0$, is uniquely determined by its arithmetic means $s_n(f)$ and $s_{0n}(f)$ over equally spaced points on $\partial D$. We also give an explicit formula for recapturing $f$ from its means $s_n(f)$ and $s_{0n}(f)$. Furthermore, we derive the relations between $s_n(f)$ and $s_{0n}(f)$ which are necessary and sufficient for the analytic continuability of $f$ from $D$ to the whole disc.

1. Introduction. Let $U : |z| < 1$ be the open unit disc and $T : |z| = 1$ be the unit circle in the complex plane. For an $\epsilon > 0$, we let $A^{1+\epsilon}(U)$ denote the class of all functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with $a_n = O(1/n^{1+\epsilon})$. If $f$ is a continuous function on $T$, we consider the arithmetic means

$$s_n(f) = \frac{1}{n} \sum_{k=1}^{n} f(w_n^k),$$

$n = 1, 2, \ldots$, of $f$ on $T$, where $w_n^k = \exp(i2\pi k/n)$ are the $n$th roots of unity. It is known (cf. [1]) that if $f \in A^{1+\epsilon}(U)$ then the sequence $\{s_n(f)\}$ uniquely determines $f$ in $A^{1+\epsilon}(U)$. Also, an explicit representation of a function $f$ in $A^{1+\epsilon}(U)$ in terms of the sequence $\{s_n(f)\}$ is given in [3]. In this paper, we establish these results for functions holomorphic in an annulus. Hence, one can explicitly recapture a function $f$, holomorphic in a simply connected or doubly connected domain $G$ and continuous on the closure of $G$, from its “means” on the boundary $\partial G$ of $G$, provided that an explicit conformal map of $G$ onto the unit disc or an annulus can be found and has a sufficiently smooth extension to $\partial G$ and that $f$ is sufficiently smooth on $\partial D$.

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Let $0<r_0<l$, and consider the annulus $D = \{z : r_0 < |z| < l\}$. For an $\varepsilon > 0$, we denote by $A^{1+\varepsilon}(D)$ the class of all functions $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ such that for $n > 0$, $a_n = O(1/n^{1+\varepsilon})$ and $a_{-n} = O(r_0^n/n^{1+\varepsilon})$. If $f$ is a function continuous on the boundary $\partial D$ of $D$, we define (cf. [2]) the Riemann coefficients of $f$ by

$$R_n(f) = s_n(f) - s_{\infty}(f) \quad \text{and} \quad R_0n(f) = s_{0n}(f) - s_{0\infty}(f),$$

where

$$s_{0n}(f) = \frac{1}{n} \sum_{k=1}^{n} f(r_0 w_k^n), \quad n = 1, 2, \ldots,$$

and

$$s_{\infty}(f) = \lim_{n \to \infty} s_n(f), \quad s_{0\infty}(f) = \lim_{n \to \infty} s_{0n}(f).$$

For all functions $f$ "smooth" on $\partial D$, it is known (cf. [2]) that the Riemann coefficients $R_n(f)$ and $R_{0n}(f)$ have similar asymptotic decay as the Fourier coefficients $a_n(f)$ and $a_{0n}(f)$ respectively, where

$$a_n(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) e^{-int} dt \quad \text{and} \quad a_{0n}(f) = \frac{1}{2\pi} \int_0^{2\pi} f(r_0 e^{it}) e^{-int} dt.$$

It is also known (cf. [8, p. 6]) that $f$ is holomorphic in $D$ if and only if $a_{0n}(f) = a_n(f) r_0^n$ for all $n = 0, \pm 1, \cdots$. On the other hand, it is easy to see that for functions $f$ holomorphic in $D$, $R_n(f)$ and $R_{0n}(f)$ are not related, since there are rational functions $q_n$ and $q_{0n}$ satisfying $R_m(q_n) = \delta_{m,n}$, $R_{0n}(q_m) = 0$, $R_m(q_{0n}) = 0$ and $R_{0m}(q_{0n}) = \delta_{m,n}$ for all $m$ and $n$. However, we will give the relations between $R_n(f)$ and $R_{0n}(f)$ which are necessary and sufficient for functions $f \in A^{1+\varepsilon}(D)$ to be of class $A^{1+\varepsilon}(U)$.

2. Uniqueness, representation and analytical continuability theorems. We first establish the following uniqueness theorem.

**Theorem 1.** Let $f \in A^{1+\varepsilon}(D)$ for some $\varepsilon > 0$ satisfy

$$(1) \quad s_n(f) = 0 \quad \text{and} \quad s_{0n}(f) = 0$$

for $n = 1, 2, \cdots$. Then $f$ is the zero function. Furthermore, for each positive integer $n$ there exist two rational functions

$q_n(z) = \sum_{k=-n}^{n} a_k z^k, \quad q_{0n}(z) = \sum_{k=-n}^{n} a_0 z^k$

with $a_0 = a_{00} = 0$ such that

$s_m(q_n) = \delta_{m,n}, \quad s_{0m}(q_n) = 0, \quad s_m(q_{0n}) = 0$ and $s_{0m}(q_{0n}) = \delta_{n,m}$ for all $m, n = 1, 2, \cdots$. 

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Proof. Since $f$ is holomorphic in $D$, we write $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ with
\[
a_0 = \frac{1}{2\pi i} \int_{|z|=1} f(z) \frac{dz}{z} = \lim_{n \to \infty} s_n(f) = 0.\]
Let $g(z) = \sum_{n=1}^{\infty} (a_n + a_{-n}) z^n$. Then $g \in A^{1+\epsilon}(U)$ and $s_n(g) = s_n(f) = 0$ for all $n=1, 2, \cdots$. Hence, we can conclude from a uniqueness theorem in [1] that $a_n + a_{-n} = 0$ for all $n$. Similarly, we also consider
\[
h(z) = \sum_{n=1}^{\infty} \left( a_n r_0^n + a_{-n} \frac{1}{r_0^n} \right) z^n,
\]
and conclude that $s_n(h) = s_0^n(f)$, $n=1, 2, \cdots$, and hence that $a_n r_0^n + a_{-n} r_0^{-n} = 0$ for all $n$. Since $0 < r_0 < 1$, it is clear that $a_n = 0$ for all $n$.

Next, we prove the existence of $q_n$. The proof of the existence of $q_0n$ is similar. Since $s_m(q_n) = s_m(q_n) = 0$ for all $m > n$, we need only consider the following two systems of $n$ equations:
\[
\begin{align*}
  s_1(q_n) &= (a_1 + a_{-1}) + \cdots + (a_n + a_{-n}) = 0 \\
  s_2(q_n) &= (a_2 + a_{-2}) + (a_4 + a_{-4}) + \cdots = 0 \\
  \vdots & \quad \vdots \\
  s_{n-1}(q_n) &= (a_{n-1} + a_{-(n-1)}) = 0 \\
  s_n(q_n) &= a_n + a_{-n} = 1;
\end{align*}
\]
\[
\begin{align*}
  s_{01}(q_n) &= (a_1 r_0 + a_{-1} r_0^{-1}) + \cdots + (a_n r_0^n + a_{-n} r_0^{-n}) = 0 \\
  s_{02}(q_n) &= (a_2 r_0^2 + a_{-2} r_0^{-2}) + (a_4 r_0^4 + a_{-4} r_0^{-4}) + \cdots = 0 \\
  \vdots & \quad \vdots \\
  s_{0,n-1}(q_n) &= (a_{n-1} r_0^{n-1} + a_{-(n-1)} r_0^{-(n-1)}) = 0 \\
  s_{0n}(q_n) &= a_n r_0^n + a_{-n} r_0^{-n} = 0.
\end{align*}
\]
Since the coefficient matrices for $(a_k + a_{-k})$ and $(a_k r_0^k + a_{-k} r_0^{-k})$ are non-singular, there are unique solutions for $(a_k + a_{-k})$ and $(a_k r_0^k + a_{-k} r_0^{-k})$, and hence for $a_k$ and $a_{-k}$, $k=1, \cdots, n$.

To establish our representation theorem, we first obtain explicit formulas for $q_n$ and $q_0n$. Let $\mu(n)$ be the Möbius function of $n$:
\[
\mu(n) = \begin{cases} 
1, & \text{if } n = 1, \\
(-1)^k, & \text{if } n = q_1 \cdots q_k, \\
0, & \text{if } p^2 | n \text{ for some } p > 1,
\end{cases}
\]
where $q_1, \cdots, q_k$ are distinct primes.
Lemma 1. For each \( n = 1, 2, \ldots \),

\[
q_n(z) = \sum_{j|n} \frac{\mu(n/j)}{r_0^{-j} - r_0^j} \left( \left( \frac{z}{r_0} \right)^j - \left( \frac{z}{r_0} \right)^{-j} \right)
\]

and

\[
q_0n(z) = \sum_{j|n} \frac{\mu(n/j)}{r_0^{-j} - r_0^j} \{ z^j - z^{-j} \}.
\]

Proof. We observe that the means

\[
s_n \left( \frac{(z/r_0)^j - (z/r_0)^{-j}}{r_0^{-j} - r_0^j} \right) = \begin{cases} 1, & \text{if } n \mid j, \\ 0, & \text{if } n \nmid j \end{cases}
\]

and

\[
s_0n \left( \frac{(z/r_0)^j - (z/r_0)^{-j}}{r_0^{-j} - r_0^j} \right) = 0 \text{ for all } n = 1, 2, \ldots .
\]

Hence, by virtue of Theorem 1, we have

\[
\frac{(z/r_0)^j - (z/r_0)^{-j}}{r_0^{-j} - r_0^j} = \sum_{n|j} q_n(z)
\]

for \( j = 1, 2, \ldots \). We now use the Möbius inversion theorem (cf. [5]) to obtain (2). The proof of (3) is similar.

Theorem 2. Let \( f \in A^{1+\epsilon}(D) \) for some \( \epsilon > 0 \). Then the series

\[
\sum_{k=1}^{\infty} R_k(f)q_k(z) + \sum_{k=1}^{\infty} R_{0k}(f)q_{0k}(z) + s_\infty(f)
\]

converges uniformly to \( f \) on \( D \) and

\[
\left| f(z) - \sum_{k=1}^{m} R_k(f)q_k(z) - \sum_{k=1}^{m} R_{0k}(f)q_{0k}(z) - s_\infty(f) \right| = O \left( \frac{1}{m^\delta} \right)
\]

uniformly on \( D \) for any fixed \( \delta, 0 < \delta < \epsilon \).

The series (4) is now called the Riemann series of the function \( f \) in \( D \) (cf. [3]).

Proof. For \( r_0 \leq |z| \leq 1 \), we have

\[
|q_k(z)| \leq \sum_{j|k} \frac{1 + r_0^{2j}}{1 - r_0^{2j}} \leq \frac{2d(k)}{1 - r_0^2}
\]

where \( d(k) \) denotes the number of divisors of \( k \). Using the well-known
estimate \(d(k) = O(k^{-\delta})\), where \(0 < \delta < \varepsilon\) (cf. [5]), and the fact that \(R_k(f) = O(1/k^{1+\varepsilon})\) and \(R_0(f) = O(r_0^{k}/k^{1+\varepsilon})\), which follows from the assumptions on \(f\) (cf. [2]), we can conclude that the series (4) converges uniformly on \(\bar{D}\) to some function \(h\), holomorphic in \(D\) and continuous on \(\partial D\). Furthermore, we have

\[
\left| h(z) - \sum_{k=1}^{m} R_k(f) q_k(z) - \sum_{k=1}^{m} R_{0k}(f) q_{0k}(z) - s_\infty(f) \right| = O\left( \frac{1}{m^\delta} \right)
\]

uniformly on \(\bar{D}\). Now, we use Lemma 1 to estimate the Fourier coefficients of \(h\): For \(m > 0\),

\[
am_m(h) = a_m \left[ \sum_{k=1}^{\infty} R_k(f) \sum_{j|k} \frac{\mu(k/j)}{r_0^{-j} - r_0^{-i}} \{(z/r_0)^j - (z/r_0)^{-j}\} \right.
\]

\[
+ \sum_{k=1}^{\infty} R_{0k}(f) \sum_{j|k} \frac{\mu(k/j)}{r_0^{-j} - r_0^{-i}} (z^j - z^{-j}) + s_\infty(f) \right] = \frac{1}{1 - r_0^{2m}} \sum_{k=1}^{\infty} R_{mk}(f) \mu(k) + \frac{r_0^{m}}{1 - r_0^{2m}} \sum_{k=1}^{\infty} R_{0, mk}(f) \mu(k) = O\left( \frac{1}{m^{1+\varepsilon}} \right).
\]

Similarly, for \(m < 0\),

\[
am_m(h) = \frac{-r_0^{-2m}}{1 - r_0^{2m}} \sum_{k=1}^{\infty} R_{-mk}(f) \mu(k) + \frac{-r_0^{-m}}{1 - r_0^{-2m}} \sum_{k=1}^{\infty} R_{0, -mk}(f) \mu(k) = O(r_0^{|m|}/|m|^{1+\varepsilon}).
\]

Hence, \(h \in A^{1+\varepsilon}(D)\) and the means of \(h\) are

\[
s_m(h) = s_m \left[ \sum_{k=1}^{\infty} R_k(f) q_k + \sum_{k=1}^{\infty} R_{0k}(f) q_{0k} + s_\infty(f) \right] = \sum_{k=1}^{\infty} R_k(f) \delta_{m,k} + s_\infty(f) = R_m(f) + s_\infty(f) = s_m(f),
\]

and similarly, \(s_{0m}(h) = R_{0m}(f) + s_\infty(f) = s_{0m}(f)\), for all \(m = 1, 2, \ldots\). Hence, \(f = h\) by Theorem 1.

For each \(n = 1, 2, \ldots\), let \(p_n(z) = \sum_{k|n} \mu(n/k) z^k\) as in [3]. We have

**Theorem 3.** Let \(f \in A^{1+\varepsilon}(D)\) for some \(\varepsilon > 0\). Then \(f\) is in \(A^{1+\varepsilon}(U)\) if and only if for all \(m \geq 1\)

\[
R_{0m}(f) = \sum_{j=1}^{\infty} p_j(r_0^m) R_{mj}(f).
\]

Here, it is clear that the series in (5) converges for every \(f\) in \(A^{1+\varepsilon}(D)\).
Proof. An easy calculation shows that

\[ R_{0k}(p_j) = p_\alpha(r_0^k) \quad \text{if } j = \alpha k \]
\[ = 0 \quad \text{if } k \not| j. \]

In [3], it is proved that if \( f \in A^{1+\epsilon}(U) \) then \( f(z) = \sum_{k=1}^{\infty} R_k(f)p_k(z) + s_\infty(f) \) uniformly in \( \overline{U} \). Hence, we have, by (6),

\[ R_{0m}(f) = \sum_{j=1}^{\infty} R_{mj}(f) \sum_{\alpha | j} \mu\left(\frac{j}{\alpha}\right) r_0^{m\alpha} \]

which is (5). To prove the converse, we first prove the following identities for all \( k \) and \( n \):

\[ \sum_{j|k} p_j(r_0^{kn/j}) \mu\left(\frac{k}{j}\right) = r_0^n \mu(k). \]

Indeed,

\[ \sum_{j|k} p_j(r_0^{kn/j}) \mu\left(\frac{k}{j}\right) = \sum_{j|k} \mu\left(\frac{k}{j}\right) \sum_{\alpha | j} \mu\left(\frac{j}{\alpha}\right) r_0^{kn/j} \]
\[ = \sum_{j|k} \mu\left(\frac{k}{j}\right) \sum_{\alpha | j} \mu(\alpha) r_0^{kn/\alpha} = \sum_{\alpha | k} \sum_{j|k/\alpha} \mu\left(\frac{k}{j}\alpha\right) \mu\left(\frac{k}{j}\right), \]

so that (7) follows from the identity \( \sum_{j|n} \mu(j) = \delta_{1,n} \).

From Theorem 2, we have

\[ f(z) = \sum_{j=1}^{\infty} R_j(f)q_j(z) + \sum_{j=1}^{\infty} R_{0j}(f)q_0j(z) + s_\infty(f) = \sum_{n=-\infty}^{\infty} a_n z^n. \]

It is clear that for each \( n > 0 \),

\[ a_n = \sum_{m=1}^{\infty} \frac{\mu(m)}{r_0^n - r_0^{-n}} R_{nm}(f)r_0^n + \sum_{m=1}^{\infty} \frac{\mu(m)R_{0,mn}(f)}{r_0^n - r_0^{-n}}. \]

Since \( R_k(f) = O(1/k^{1+\epsilon}) \), we obtain, by (5) and (7),

\[ (r_0^n - r_0^{-n})a_n = \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} p_j(r_0^{nm})R_{jm}(f) - \sum_{m=1}^{\infty} r_0^n R_{nm}(f) \mu(m) \]
\[ = \sum_{k=1}^{\infty} R_{kn} \left( \sum_{j|k} p_j(r_0^{kn/j}) \mu\left(\frac{k}{j}\right) - r_0^n \mu(k) \right) = 0. \]

3. Final remark. The results in this paper are generalizations of those studied in ([1], [3]). Recently, Patil ([6], [7]) has given an explicit representation of an \( H^p \) function in terms of its boundary values on a small subset \( S \) of the unit circle. It is, therefore, also interesting to know whether or not just the arithmetic means of the values of a function \( f \in A^{1+\epsilon}(U) \) at
points "equally spaced" on $S$ would uniquely determine $f$, and if so, whether or not an "explicit" formula for recapturing $f$ from these means could be given. If $S$ is an arc, some results have been recently obtained in [4].

REFERENCES


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