EXPLICIT CONDITIONS FOR THE FACTORIZATION OF nTH ORDER LINEAR DIFFERENTIAL OPERATORS

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Abstract. For any integer k with $1 \leq k \leq n$ sufficient conditions on the coefficients $p_i$ are given for the factorization of certain classes of operators $L = p_n y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_0 y$ into a product $L = PQ$ where $P$ and $Q$ are operators of the same type of orders $n-k$ and $k$, respectively. A special case yields that if $(-1)^n p_0 \geq 0$ then $y^{(n)} + p_0 y$ is factorable into a product of two regular differential operators of orders $n-k$ and $k$.

1. Introduction. We consider the classical nth order regular linear differential operator

$$L y = p_n y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_0 y$$

where $p_i$ for $i = 0, 1, \cdots, n$ are continuous real valued functions with $p_n(t) \neq 0$ on some interval $[a, b)$ for $-\infty < a < b \leq \infty$.

As a consequence of our main theorem we obtain the following factorization results: Suppose $p_n(t) \equiv 1$, $p_i \equiv 0$ for $i = 1, 2, \cdots, n-3$ and $p_{n-2} \geq 0$.

(a) If $p_0 \geq 0$, then $L$ has a right factor of any even order, i.e., for any even positive integer $k < n$ there exist operators $P$ and $Q$ of type (1.1) and orders $n-k$ and $k$, respectively, such that $L = PQ$. (b) If $p_0 \leq 0$, then $L$ has a right factor of any odd order.

Right factors $Q$ of order $n-1$ are obtained under a much weaker hypothesis. For the case when the order of $Q$ is 1 stronger results are well known. Some extensions of these results are indicated as well as a generalization to a quasi-differential operator. Also a couple of applications to boundary value problems are given.

According to a well-known result of Pólya [12] the operator $L$ has a factorization into "products" of first order operators

$$L y = r_n (r_{n-1} \cdots (r_1 (r_0 y)' \cdots )')$$

if and only if the equation $L y = 0$ has a fundamental set of solutions

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\[ y_1, \ldots, y_n \text{ such that} \]

\[(1.3) \quad W_k > 0 \quad \text{for} \quad k = 1, \ldots, n-1 \]

where \( W_1 = y_1 \) and \( W_k = \det_{i,j=1,\ldots,k}[y_j^{(i-1)}] \) for \( k = 2, \ldots, n-1 \). For a short and elegant proof of this factorization see [13]. A factorization of type (1.2) on an interval \((a, b)\) is known to be equivalent to disconjugacy on \((a, b)\). The problem of finding explicit conditions on the coefficients \( p_i \) which yield a factorization of type (1.2) or, equivalently, assure that \( L \) is disconjugate has received considerable attention. For some recent papers see [1], [6], [11], [14], [15].

In [17] it is shown that an operator \( L \) of type (1.1) has a factorization

\[(1.4) \quad L = PQ \]

where \( P \) and \( Q \) are of type (1.1) of orders \( n-k \) and \( k \), respectively, if and only if there exist \( k \) linearly independent solutions of \( Ly=0 \) whose Wronskian \( W_k \) satisfies

\[(1.5) \quad W_k > 0. \]

Not much seems to be known about conditions which yield factorizations of type (1.4). Some conditions—involving the Lagrange bilinear form for solutions—which imply a factorization of some \( 2n \)th order operators as products of two \( n \)th order ones were obtained by Rellich and Heinz in [4]. Direct conditions on the coefficients which yield factorizations of types (1.2) and (1.4) are obtained in [16] for the case \( n=3 \).

2. We use the notation \( X^\geq 0 \) for a matrix or vector \( X \) to mean that each component of \( X \) is nonnegative. Similarly \( X > 0 \) means each component is strictly positive.

Our development is based on the following two lemmas. The first one is a very useful result due to Mikusinski [9]. The second one is the result stated above from [17].

**Lemma 1.** Let \( y_i = \sum_{j=1}^{m} F_{ij} y_j \) for \( i = 1, \ldots, m \) be a system of differential equations with real valued continuous coefficients which are nonnegative for \( i \neq j \) on \([a, b)\). If \( Y = (y_i) \) for \( i = 1, \ldots, m \) is a solution vector satisfying \( Y(a)^\geq 0 \), then \( Y(t)^\geq 0 \) for \( t \) in \([a, b)\). Moreover if some component \( y_i \) is positive at a point \( c \) in \([a, b)\), then \( y_i(t) \) is positive for \( t > c \).

**Lemma 2.** A necessary and sufficient condition that a differential operator \( L \) of type (1.1) has a factorization (1.4) where \( P \) and \( Q \) are operators of type (1.1) of orders \( n-k \) and \( k \) respectively is that there exist \( k \) linearly independent solutions of \( Ly=0 \) whose Wronskian \( W_k \) is positive.

For convenience of notation we denote the solution space of \( Ly=0 \) by \( S \). In the rest of the paper we assume, for convenience, that \( p_n(t)^\equiv 1 \).
As a consequence of Lemma 1 and the classical vector matrix representation of the equation \(Ly=0\) we obtain

**Theorem 1.** Suppose \(p_i \leq 0\) for \(i=0, 1, \ldots, n-2\). If \(y \in S\) with \(y^{(i)}(a) \geq 0\) for \(i=0, 1, \ldots, n-1\) and \(y^{(r)}(a) > 0\) for \(r=0, \ldots, n-1\) and \(t > a\) and \(y^{(p)}(t) > 0\) for \(p=0, \ldots, r\) and \(t > a\).

Using positive initial conditions at \(a\) to determine a solution of \(Ly=0\), then as a consequence of Theorem 1 and Lemma 2 we obtain

**Corollary 1.** If \(p_i \leq 0\) for \(i=0, 1, \ldots, n-2\); then the operator \(L\) can be factored: \(L = PQ\) where \(P\) is of order \(n-1\) and \(Q\) is of order 1.

Theorem 1 is stated only for the sake of completeness since the result is known—although many authors use a sign condition on \(p_{n-1}\).

The conclusion of Corollary 1 still holds if the signs of the coefficients alternate—see [2, p. 508].

**Theorem 2.** If \((-1)^{n-1}p_i \leq 0\) for \(i=0, 1, \ldots, n-2\) then there exists \(y \in S\) with \(y(t) > 0\) for \(t > a\).

Let \(z_1, \ldots, z_n\) be solutions of \(Ly=0\) determined by the initial conditions

\[
(2.2) \quad z_j^{(i-1)}(a) = \delta_{ij} \quad \text{for } i, j = 1, \ldots, n.
\]

Our main result is:

**Theorem 3.** Suppose \(p_i \equiv 0\) for \(i=1, 2, \ldots, n-3\) and \(p_{n-2} \leq 0\). Let \(W_k\) denote any \(k\)th order Wronskian \(W(z_1, z_2, \ldots, z_{k-1})\) for \(i=1, 2, \ldots, n-k+1, k=1, \ldots, n-1\). If \(p_0 \geq 0\) and \(k\) is even, then \(W_k > 0\) on \((a, b)\). If \(p_0 \leq 0\) and \(k\) is odd, then \(W_k > 0\) on \((a, b)\).

**Proof.** Let \(y_1, \ldots, y_k\) be any solutions of \(Ly=0\). For integers \(i_j \leq n, j=1, \ldots, k\), we define

\[
D(i_1, i_2, \ldots, i_k) = \det \begin{vmatrix}
y_1^{i_1}, & y_2^{i_2}, & \ldots, & y_k^{i_k} \\
y_1^{i_2}, & y_2^{i_2}, & \ldots, & y_k^{i_k} \\
\vdots & \vdots & \ddots & \vdots \\
y_1^{i_k}, & y_2^{i_k}, & \ldots, & y_k^{i_k}
\end{vmatrix}
\]

Note that \(D(i_1, \ldots, i_k) = 0\) if any pair of \(i_j\)'s are equal and that the \(i_j\)'s can always be put in increasing order by a change in sign, if necessary. Also note that \(D(i_1, i_2, \ldots, i_{k-1}, n)\) can be expressed in terms of determinants involving only derivatives of orders less than \(n\) by replacing \(y^{(n)}_i\) by \(-p_{n-1}y^{(n-1)}_i - \cdots - p_0y_i\).
If $i_k < n - 1$ observe that

$$D'(i_1, i_2, \ldots, i_k) = D(i_1 + 1, i_2, \ldots, i_k) + D(i_1, i_2 + 1, i_3, \ldots, i_k) + \cdots + D(i_1, i_2, \ldots, i_k + 1).$$

If $i_k = n - 1$, then

$$D'(i_1, \ldots, i_k) = D(i_1 + 1, i_2, \ldots, i_k) + D(i_1, i_2 + 1, i_3, \ldots, i_k) + \cdots + D(i_1, i_2, \ldots, i_{k-1} + 1, i_k) + D(i_1, i_2, \ldots, i_{k-1}, i_k + 1) = n$$

and

$$D(i_1, i_2, \ldots, i_{k-1}, n) = -p_0 D(i_1, i_2, \ldots, i_{k-1}, 0) - p_1 D(i_1, i_2, \ldots, i_{k-1}, 1) - p_2 D(i_1, i_2, \ldots, i_{k-1}, 2) - \cdots - p_{n-1} D(i_1, i_2, \ldots, i_{k-1}, n-1).$$

From this it follows that the set of $D(i_1, i_2, \ldots, i_k)$ for $i_1 < i_2 < \cdots < i_k$, $i_j = 0, 1, \ldots, n - 1, j = 1, \ldots, k$, are solutions of a system of differential equations $Y' = FY$ where $Y$ is a vector of order $(k)$ with components $D(i_1, i_2, \ldots, i_k)$ and $F$ is an $(k)$ by $(k)$ matrix whose entries consist of 0's and 1's and $(-1)_0 p_0, +p_1, -p_1, +p_2, -p_2, \ldots, +p_{n-3}, -p_{n-3}, -p_{n-2}, -p_{n-1}$. The components of $Y$ are ordered as follows:

$$Y = \text{transpose of} [D(0, 1, \cdots, k-1), D(0, 1, \cdots, k-2, k), \ldots, D(0, 1, \ldots, k-3, k-1, k), \ldots, D(0, 1, \cdots, k-3, k-1, n-1), \ldots].$$

Note that $-p_{n-1}$ is on the diagonal of $F$—hence no sign condition is needed for $p_{n-1}$ in order to use Lemma 1.

Therefore by Lemma 1, all the components of $Y$ are nonnegative on $[a, b]$ if $Y(a) \geq 0$.

For any $i = 1, 2, \ldots, n-k+1$ let $y_1 = z_i, y_2 = z_{i+1}, \ldots, y_k = z_{i+k-1}$ where the $z_j$'s are the solutions determined by the initial conditions $(2.2)$.

For such a choice of $y_1, y_2, \ldots, y_k$ the initial vector $Y(a) \geq 0$ and the component $D(i-1, i, i+1, \ldots, i+k-2)$ has the value 1 at $a$. The first component of $Y$, namely $W_k$, is nondecreasing on $[a, b)$ since $W_k = D(0, 1, \cdots, k-2, k-1) = D(0, 1, \cdots, k-2, k) \geq 0$. Hence $W_k$ is positive on $(a, b)$ since $W_k$ identically zero on some open interval $(a, t_0)$ would imply that all the components of $Y$ are identically zero on $(a, t_0)$. But the component $D(i-1, i, \cdots, i+k-2)$ cannot be zero in $(a, t_0)$ since it is positive at $a$ and continuous. This completes the proof of Theorem 3.

The factorization result mentioned in the introduction is obtained by
taking \( i=1 \) in the above argument and noting that \( W(z_1, z_2, \ldots, z_k)(a) = 1 \) to get \( W(z_1, z_2, \ldots, z_k)(t) > 0 \) for \( t \geq a \). The rest follows from Lemma 2.

We remark that, as shown in [17], the right factor \( Q \) can be taken as \( Qy = W(y_1, y_2, \ldots, y_k, y) \) where \( y_1, y_2, \ldots, y_k \) is any set of solutions of \( Ly = 0 \) satisfying \( W_k = W(y_1, \ldots, y_k) > 0 \).

For the case \( k = n - 1 \), we obtain a stronger result:

**Theorem 4.** Suppose \((-1)^{n+1}p \geq 0\) for \( j = 0, 1, \ldots, n - 2 \). Then there exist \( y_1, y_2, \ldots, y_{n-1} \in S \) such that \( W_{n-1} = W(y_1, y_2, \ldots, y_{n-1}) > 0 \).

**Proof.** The proof is similar to that of Theorem 3, therefore we merely outline it here.

Let solutions \( y_i \) of \( Ly = 0 \) be determined by initial conditions \( y_i^{j-1}(a) = \delta_{ij} \) for \( i = 1, \ldots, n - 1 \) and \( j = 1, \ldots, n \). Define \( D(i_1, i_2, \ldots, i_{n-1}) \) as in the proof of Theorem 3. As in the proof of Theorem 3 we then show that the column vector

\[
Y = \begin{bmatrix}
D(0, 1, 2, \ldots, n - 2), D(0, 1, 2, \ldots, n - 3, n - 1), \\
D(0, 1, 2, \ldots, n - 4, n - 2, n - 1), \\
\vdots \\
D(0, 2, 3, \ldots, n - 1), D(1, 2, 3, \ldots, n - 1)
\end{bmatrix}
\]

satisfies the differential system \( Y' = FY \) where \( F \) is the matrix having 1’s on the super diagonal, \( -p_{n-1} \) on the diagonal except for the \((1, 1)\) and \((n, n)\) position, 0’s elsewhere except for the first column which is, from top to bottom, \([0, -p_{n-2}, +p_{n-3}, \ldots, (-1)^{n-1}p_1, (-1)^{n+1}p_0]\). Noting that \( Y(a) = [1, 0, \ldots, 0] \), we conclude, by Lemma 1, that

\[
y_1(t) = D(0, 1, 2, \ldots, n - 2)(t) = W_{n-1}(t) = W(y_1, y_2, \ldots, y_{n-1})(t) > 0 \quad \text{for } t \geq a.
\]

As an immediate consequence of Theorem 4 and Lemma 2 we obtain

**Corollary 2.** Under the hypothesis of Theorem 4, \( L \) has a right factor \( Q \) of order \( n - 1 \).

3. Here we indicate some extensions of these results and give a couple of applications.

**Remark 1.** This is a result obtained by Kim in [5]. Suppose \( u, v \) are solutions of \( Ly = 0 \) satisfying \( W_2 = W(u, v) = uv' - vu' > 0 \) on \([a, b]\). By Lemma 2 we have a factorization \( L = PQ \) where \( Q \) has order 2 and by the remark following the proof of Theorem 3 we can take \( u, v \) to be a fundamental set of solutions of \( Qy = 0 \). Therefore we can conclude such things as:

(i) Neither \( u \) nor \( v \) can have a double zero on \([a, b]\).
(ii) \( u^j, v^j \) have no common zero on \([a, b]\) for \( j = 0, 1 \).
(iii) Between any two zeros of one there is a zero of the other.
Remark 2. If \( p_i \in \mathcal{C}^i_{(a,b)} \) for \( i=0, 1, \cdots, n \) then the classical adjoint operator \( L^+ \), defined by

\[
L^+ y = (-1)^n (p_n y)^{(n)} + (-1)^{(n-1)} (p_{n-1} y)^{(n-1)} + \cdots + p_0 y
\]

can be put into the form (1.1). By applying the above factorization results to the adjoint operator \( L^+ \) and using the fact that \( L = PQ \) if and only if \( L^+ = Q^+P^+ \)—see [10]—additional sufficient conditions for factorization can be obtained. We illustrate with an example: Consider the fourth order operators \( L \) and \( L^+ \) defined by

\[
Ly = y^{(4)} + p_3 y^{(3)} + p_2 y^{(2)} + p_1 y^{(1)} + p_0 y,
\]

\[
L^+ y = y^{(4)} - (p_3 y)^{(3)} + (p_2 y)^{(2)} - (p_1 y)' + p_0 y
\]

\[
\]

Applying our factorization theorems we have

**Corollary 3.** If \( p_2 - 3p_3 \leq 0, 2p_2 - p_1 - 3p_3 \leq 0, p_0 - p_1 + p_3 - p_2 \leq 0, \) then \( L^+ = P_3 Q_1 \) where \( P_3, Q_1 \) are operators of type (1.1) of orders 3 and 1, respectively. Hence \( L = Q_1^+ P_3^+ \).

**Corollary 4.** If \( p_0 - p_1 + p_3 - p_2 \leq 0, 2p_2 - p_1 - 3p_3 \geq 0 \) and \( p_2 - 3p_3 \leq 0, \) then \( L = Q_1^+ P_3^+ \) where \( Q_1, P_3 \) are of orders 1 and 3, respectively.

**Corollary 5.** If \( p_0 - p_1 + p_3 - p_2 \leq 0, 2p_2 - p_1 - 3p_3 = 0 \) and \( p_2 - 3p_3 \leq 0, \) then \( L = P_2 Q_2 \) and \( L^+ = Q_2^+ P_2^+ \) where \( P_2, Q_2 \) are operators of type (1.1) of order 2.

**Corollary 6.** If \( p_0 - p_1 + p_3 - p_2 \leq 0, 2p_2 - p_1 - 3p_3 \geq 0 \) and \( p_2 - 3p_3 \leq 0, \) then there exist operators \( P_1 \) and \( Q_3 \) of type (1.1) of orders 1 and 3, respectively such that \( L^+ = P_1 Q_3 \) and hence \( L = Q_3^+ P_1^+ \).

As another application of some of these factorizations we list

**Theorem 5.** Under the hypothesis of Theorem 4, the boundary value problem

\[
Ly = 0, \quad y(a) = 0, \quad y(\beta) = y'(\beta) = \cdots = y^{(n-2)}(\beta) = 0
\]

for any \( a, \beta \) in \( [a, b) \) with \( a < \beta \) has no nontrivial solution.

**Proof.** Suppose \( y \) is a nontrivial solution. Let \( c \) be the first point to the left of \( \beta \) such that \( y^{(i)}(c) \neq 0 \) for \( i = 0, \cdots, n-2 \) and \( y^{(n-1)}(c) = 0. \)
Determine solutions $y_1, y_2, \cdots, y_{n-2}$ by the initial conditions:

$y_1(c) = y(c), \quad y'_1(c) = 0, \quad \cdots, \quad y_{n-1}^{(n-1)}(c) = 0$

$y_2(c) = 0, \quad y'_2(c) = y'(c), \quad y'_2(c) = 0, \quad \cdots, \quad y_{n-1}^{(n-1)}(c) = 0$

$\quad \vdots$

$y_{n-2}(c) = 0, \quad \cdots, \quad y_{n-2}^{(n-3)}(c) = 0, \quad y_{n-2}^{(n-2)}(c) = y_{n-2}^{(n-3)}(c),$ $\quad y_{n-2}^{(n-2)}(c) = 0, \quad y_{n-1}^{(n-2)}(c) = 0.$

Define $D(i_1, i_2, \cdots, i_{n-1})$ as in the proof of Theorem 3 using $y = y_{n-1}$ and note that $D(0, 1, 2, \cdots, n-2)(c) = W(y_1, y_2, \cdots, y_{n-2}, y)(c) = y(c)y'(c) \cdots y_{n-2}^{(n-2)}(c) \neq 0$ and all other $D(i_1, i_2, \cdots, i_{n-1})$ are zero at $c$.

Repeated applications of the mean value theorem show that the signs of $y(c), y'(c), y''(c), \cdots, y_{n-2}^{(n-2)}(c)$ alternate. Here we are using the fact that $y^{(i-1)}(c) \neq 0$ and that $c$ is the first point to the left of $\beta$ such that $y^{(i)}(c) \neq 0$ for $i = 0, \cdots, n-2$ and $y^{(n-1)}(c) = 0$. By replacing $y$ with $-y$, if necessary, we can get the product $y(c)y'(c) \cdots y_{n-2}^{(n-2)}(c)$ positive. Proceeding as in the proof of Theorem 4 we get to the conclusion

$$W(t) = W(y_1, y_2, \cdots, y_{n-2}, y)(t) > 0 \quad \text{for } t > c.$$  

But this contradicts $W(\beta) = 0$.

We list here a couple of illustrations of Theorem 3.

**Theorem 6.** Under the hypothesis of Theorem 3, if $y$ is a nontrivial solution of $Ly = 0$ on $[a, b]$ which has a zero of order $k$ at $a$ and a zero of order $n-k$ at $c$, $a < c < b$ then $n-k$ is even if $p_0 \geq 0$ and $n-k$ is odd if $p_0 \leq 0$.

**Proof.** Such a solution $y$ can be expressed as $y = \alpha_{k+1}z_{k+1} + \cdots + \alpha_nz_n$. A zero of order $n-k$ at $c > a$ would imply $W(z_{k+1}, \cdots, z_n)(c) = 0$. A. Ju. Levin [7], [8] obtained, by different methods, this result for the operator $y^n + p_0y$.

A consequence of Theorem 6 is that, under the hypothesis of Theorem 3, no nontrivial solution of $Ly = 0$ can have zeros at $a, c$ with $a < c < b$ of combined order $> n$, because this would imply that two Wronskians of consecutive integral order are zero at $c$. But one of these has to be even and one odd.

If an operator is given in quasi-differential form—such as $(py^n)' + qy$—one can sometimes get simpler conditions by using the techniques of proof above and appropriate “quasi-derivatives” than by “stringing out” the expression into the form (1.1) and then using Theorems 1, 2, 3, and 4.
As an illustration we consider the operator

\[(3.1) \quad L_y = (p y''')'' + q y\]

where \(p \in C^2([a, b]), q \in C([a, b])\) and \(p > 0\) on \([a, b)\).

**Theorem 7.** (a) If \(q \leq 0\), then there exists a positive solution of \(L_y = 0\).
(b) If \(q \geq 0\) and \((pq)' \geq 0\), then there exist solutions \(u, v\) of \(L_y = 0\) such that \(W_z = W(u, v) > 0\). Hence \(L\) has a factorization into a product of two second order operators.

**Proof.** Part (a). The proof is similar to that of Theorem 1. The main modification is that the vector \(Y\) used here is \(Y = \text{column vector} [y, y', py'', (py''')']\) and the resulting matrix \(F\) has components all zero except for \(-q\) in the (4, 1) position, 1 in the (1, 2) and (3, 4) positions and \(1/p\) in the (2, 3) position. The details are omitted.

Part (b). Determine solutions \(u, v\) of \(L_y = 0\) by the initial conditions:

\[
\begin{align*}
&u(a) = 1, \quad u'(a) = 0, \quad u''(a) = 0, \quad u'''(a) = 0, \\
&v(a) = 0, \quad v'(a) = 1, \quad v''(a) = 0, \quad v'''(a) = 0.
\end{align*}
\]

Let \(z = W(u, v) = uv' - u'v\). We show that the column vector \(Y = [z, pz', (pz')', (pz'')', (pz''')']\) satisfies a differential system \(Y' = FY\): Note that

\[
\begin{align*}
z' &= \begin{bmatrix} u \\ v \\ u'' \\ v'' \end{bmatrix}, \\
(pz')' &= \begin{bmatrix} u' \\ v' \\ pu'' \\ pv'' \end{bmatrix} + \begin{bmatrix} u \\ v \\ (pu')' \\ (pv')' \end{bmatrix}, \\
(pz')'' &= 2 \begin{bmatrix} u' \\ v' \\ (pu'')' \\ (pv'')' \end{bmatrix}, \\
(pz')''' &= 2 \begin{bmatrix} u'' \\ v'' \\ (pu''')' \\ (pv''')' \end{bmatrix} + 2qz
\end{align*}
\]

where we substituted \(-qu\) for \((pu'')'\) and \(-qv\) for \((pv'')'\) in \([p(pz')']' = 4q(pz') + 2(pq)'z\).

From these computations we see that \(Y' = FY\) where \(F\) is the matrix with components zero everywhere except for 1's in the (2, 3) and (3, 4) positions, \(1/p\) in the (1, 2) and (4, 5) positions, \(+2(pq)\) in the (5, 1) position and \(+rq\) in the (5, 2) spot. The conclusion follows from Lemmas 1 and 2.
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