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PROJECTION CONSTANTS FOR $C(S)$ SPACES WITH THE SEPARABLE PROJECTION PROPERTY

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ABSTRACT. It is shown that if n and k are positive integers and $C(\omega^n k)$ is the Banach space of continuous functions on the compact set $\Gamma(\omega^n k)=\{\alpha|\alpha$ is an ordinal, $\alpha\leqq\omega^n k\}$ then $C(\omega^n k)\in P'$ if and only if $\gamma\leqq 2n+1$. This establishes the value of the projection constant for all $C(S)$ spaces possessing the separable projection property.

1. **Introduction.** A separable Banach space X has the *separable projection property* if for every separable Banach space Y and every isometric embedding $u:X\rightarrow Y$, there is a projection Π of Y onto $u(X)$. If Π can always be selected with $\|\Pi\|\leqq\lambda$, X is a P_λ' *space* (denoted $X\in P_\lambda'$). The space X has the *separable extension property* if for each separable Banach space Y with $X\subset Y$ and each isomorphism u of X into some Banach space B, there is an extension $\bar{u}$, of u from Y into B. In **[6]**, D. Dean showed that if X has the separable projection property, it is a P_λ' space for some finite λ, and this property is equivalent to the separable projection property. D. Amir (see **[1]**, **[2]**) has shown that if S is a compact metric space, then $C(S)$ has the separable projection property if and only if S is homeomorphic to the set $\Gamma(\omega^n k)$ of ordinals for some positive integers n and k.

If a Banach space X has the separable projection property, the number

$$p_s(X)=\inf\{\lambda\geqq 1 \mid X\in P_\lambda'\}$$

will be called the (separable) *projection constant*. In **[12]**, A. Sobczyk established $p_s(c_0)=2$ and R. McWilliams in **[10]** showed that $p_s(c)=3$. Recently, A. Pełczyński **[11**, p. 74**]** indicated it would be interesting to know the values of $p_s(C(\omega^n))$ for $1\leqq n<\omega$. Here we show $p_s(C(\omega^n k))=2n+1$ for $1\leqq n$, $k<\omega$. This establishes the values of the projection constant for all continuous function spaces with the separable projection property and includes McWilliam's result.

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2. **Preliminaries.** If X is a topological space, a *decomposition* D of X is a disjoint collection of closed subsets of X such that $X=\bigcup\{A : A \in D\}$. The notation X/D denotes the set D with its quotient topology. A set $A \in D$ is called *plural* if it contains at least two elements. Also, a set $A \in D$ is called a *limit set* if each open set containing B has nontrivial intersection with a plural set in $D\sim\{B\}$. The nth *derived decomposition* $D^{(n)}$ of X is defined as follows: $D^{(1)}$ is the decomposition of X consisting of the plural limit sets in D and singleton sets. Inductively, if $D^{(n)}$ is defined, then $D^{(n+1)}$ is the decomposition of X consisting of the plural limit sets in $D^{(n)}$ and singleton sets. If $D^{(n)}$ contains no plural sets, we write $D^{(n)}=0$. The concept of the nth derived set is due to R. Arens [**3**]. A subset Y of X is *D-saturated* if it is a union of sets in D. Any additional terminology and the basic properties of decompositions used here may be found, for example, in [**8**].

If X and Y are compact Hausdorff spaces and ϕ is a (continuous) map of X onto Y, then ϕ° denotes the isometric isomorphism from $C(Y)$ into $C(X)$ that takes f to $f\circ\phi$. If $Y=X/D$ and ϕ is the quotient map of D, then $\phi^\circ[C(X/D)]$ is identified with $C(X/D)$ and consists of all functions in $C(X)$ which are constant on each set in D. If λ is an ordinal, the *topological derivative* of order λ of X, denoted $X^{(\lambda)}$, is defined as follows: $X^{(0)}=X$, $X^{(\lambda+1)}=(X^{(\lambda)})'$, and $X^{(\lambda)}=\bigcap_{\beta<\lambda} X^{(\beta)}$ for λ a limit ordinal, where X' denotes the derived set of X.

3. **Results.**

LEMMA. *If H is a subspace of $C([0, 1])$ isometrically isomorphic to $C(\omega^n k)$, then there is a projection $\Pi : C([0, 1])\to H$ with $\|\Pi\|\leqq 2n+1$.*

PROOF. Let u be an isometric isomorphism from $C(\omega^n k)$ onto H. By a theorem of W. Holsztyński (see [7] or [**11**]), there is a closed subset Q of $[0, 1]$, a map ϕ of Q onto $\Gamma(\omega^n k)$ and ε in $C(Q)$ such that $|\varepsilon(q)|=1$ and $\varepsilon(q)(ug)(q)=\phi^\circ g(q)$ for all $q\in Q$ and $g\in C(\omega^n k)$. Let D be the upper semicontinuous decomposition $\{\phi^{-1}(\alpha)|\alpha\in\Gamma(\omega^n k)\}$ of Q induced by the closed map ϕ. Define $Q_1=\phi^{-1}[0, \omega^n]$ and $Q_i=\phi^{-1}(\omega^n(i-1), \omega^n i]$ for $1\leqq i<k$ and let D_i be the restriction of D to Q_i. Then D_i is an u.s.c. decomposition of the compact set Q_i. If H_i is the u.s.c. decomposition $D_i^{(1)}$ of Q_i, then $(\omega^n(i-1), \omega^n i]^{(n+1)}=\varnothing$ implies $H_i^{(n)}=0$. By Theorem 1.9 in [**4**], there is a projection P_i of $C(Q_i)$ onto $C(Q_i/H_i)$ with $\|P_i\|\leqq 2n+1$. Let Y_i be the union of the plural sets in D_i-H_i. Each plural set S in D_i-H_i is open and closed in Q_i. Let $x(S)$ be a point in S and for $f\in C(Q_i)$ define

$$\begin{aligned} P_i^* f(x) &= P_i f(x), && \text{if } x\in Q_i - Y_i \\ &= P_i f(x(S)), && \text{if } x\in S\subset Y_i. \end{aligned}$$

We show $P_i^* f \in C(Q_i/D_i)$. Since $P_i^* f$ is defined on Q_i and constant on each set in D_i, it suffices to show $P_i^* f$ is continuous.

Suppose $x_n \in Q_i$ and $x_n \to x$. If $x_n \in Q_i - Y_i$ for each n, then $x \in Q_i - Y_i$ and $P_i^* f(x_n) \to P_i^* f(x)$ since $P_i^* f$ agrees with $P_i f$ on $Q_i - Y_i$. Thus, it suffices to suppose each $x_n \in Y_i$. If $x \in Y_i$, then $x \in S$ for some open set S in D and there exists $N > 0$ such that $x_n \in S$ for $n \geqq N$. Then $P_i^* f(x_n) = P_i^* f(x)$ for $n \geqq N$; so $P_i^* f(x_n) \to P_i^* f(x)$. Therefore, we may assume $x \in Q_i - Y_i$. Let $\delta > 0$ and choose $S \in H_i$ with $x \in S$. Then $P_i f$ is constant on S. Thus there exists a D_i-saturated neighborhood U of S such that $|P_i f(y) - P_i f(x)| < \delta$ for all $y \in U$. There exists $N > 0$ such that $x_n \in U$ for all $n \geqq N$. Therefore, if $x_n \in S_n \in D_i$, then $S_n \subset U$ and

$$|P_i^* f(x_n) - P_i^* f(x)| = |P_i f(x(S_n)) - P_i f(x)| < \delta$$

for all $n \geqq N$. Consequently, $P_i^* f$ is continuous. It follows that P_i^* is a projection of $C(Q_i)$ onto $C(Q_i/D_i)$ with $\|P_i^*\| \leqq \|P_i\| \leqq 2n+1$.

Since each Q_i is both open and closed, $C(Q) = C(Q_1) \oplus C(Q_2) \oplus \cdots \oplus C(Q_k)$. For each $f = f_1 + f_2 + \cdots + f_k \in C(Q)$ with $f_i \in C(Q_i)$, let $Pf = P_1 f_1 + P_2 f_2 + \cdots + P_k f_k$. Then P is a projection of $C(Q)$ onto $C(Q/D)$ with $\|P\| \leqq 2n+1$. Let R denote the restriction operator from $C(X)$ onto $C(Q)$ and $\varepsilon' = 1/\varepsilon$. Define $\Pi = u(\phi^\circ)^{-1} P T_\varepsilon R$ where $T_\varepsilon : C(Q) \to C(Q)$ by $T_\varepsilon f = \varepsilon f$. Clearly Π is a continuous linear operator from $C(X)$ into H. If $f \in H$, say $f = ug$, then $RF = f|Q = \varepsilon' \phi^\circ g$. Therefore,

$$\Pi f = u(\phi^\circ)^{-1} P T_\varepsilon (\varepsilon' \phi^\circ g) = u(\phi^\circ)^{-1} P \phi^\circ g = ug = f.$$

This shows Π is a projection. Clearly, $\|\Pi\| \leqq 2n+1$.

THEOREM. $p_s(C(\omega^n k)) = 2n+1$.

PROOF. Let E be a separable Banach space, $S = \Gamma(\omega^n k)$, and $u : C(S) \to E$ be an isometric embedding. By the Banach-Mazur theorem, we may assume E is a subspace of $C([0, 1])$. By the preceding lemma, there is a projection Π of $C([0, 1])$ onto $u[C(S)]$. The restriction P of Π to E is a projection of E onto $u[C(S)]$ with $\|P\| \leqq 2n+1$; hence, $C(S) \in P'_{2n+1}$.

The fact that $C(S) \notin P'_\lambda$ for $\lambda < 2n+1$ is established by Amir in the proof of the theorem in [**1**]. For completeness sake, we sketch his proof. Let Σ be a countable field of subsets of S which contains a basis for the open sets in S and is closed under complements, finite unions, and the closure operation. Denote by $B(S, \Sigma)$ the closed subspace spanned in $m(S)$ by the characteristic functions of the sets of Σ. Then $B(S, \Sigma)$ is a separable Banach space containing $C(S)$. Since for each n-tuple $(k_1, k_2, \cdots, k_n)$ of positive integers, S is $(k_1, k_2, \cdots, k_n)$-Σ-connected (see [**1**] for definition) at ω^n, it follows from the lemma in [**1**] that if P is a projection of $B(S, \Sigma)$ onto $C(S)$, then $\|P\| \geqq 2n+1$. Therefore, $C(S) \notin P'_\lambda$ for $\lambda < 2n+1$.

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