PROJECTION CONSTANTS FOR C(S) SPACES WITH THE SEPARABLE PROJECTION PROPERTY

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Abstract. It is shown that if $n$ and $k$ are positive integers and $C(\omega^n k)$ is the Banach space of continuous functions on the compact set $\Gamma(\omega^n k) = \{ \alpha | \alpha \text{ is an ordinal}, \alpha \leq \omega^n k \}$ then $C(\omega^n k) \in P'$ if and only if $n \leq 2n + 1$. This establishes the value of the projection constant for all $C(S)$ spaces possessing the separable projection property.

1. Introduction. A separable Banach space $X$ has the separable projection property if for every separable Banach space $Y$ and every isometric embedding $u : X \to Y$, there is a projection $\Pi$ of $Y$ onto $u(X)$. If $\Pi$ can always be selected with $\| \Pi \| \leq \lambda$, $X$ is a $P_\lambda'$ space (denoted $X \in P_\lambda'$). The space $X$ has the separable extension property if for each separable Banach space $Y$ with $X \subset Y$ and each isomorphism $u$ of $X$ into some Banach space $B$, there is an extension $\tilde{u}$, of $u$ from $Y$ into $B$. In [6], D. Dean showed that if $X$ has the separable projection property, it is a $P_\lambda'$ space for some finite $\lambda$, and this property is equivalent to the separable projection property. D. Amir (see [1], [2]) has shown that if $S$ is a compact metric space, then $C(S)$ has the separable projection property if and only if $S$ is homeomorphic to the set $\Gamma(\omega^n k)$ of ordinals for some positive integers $n$ and $k$.

If a Banach space $X$ has the separable projection property, the number

$$p_s(X) = \inf \{ \lambda \geq 1 | X \in P_\lambda' \}$$

will be called the (separable) projection constant. In [12], A. Sobczyk established $p_s(c_0) = 2$ and R. McWilliams in [10] showed that $p_s(c) = 3$. Recently, A. Pełczyński [11, p. 74] indicated it would be interesting to know the values of $p_s(C(\omega^n))$ for $1 \leq n < \omega$. Here we show $p_s(C(\omega^n k)) = 2n + 1$ for $1 \leq n, k < \omega$. This establishes the values of the projection constant for all continuous function spaces with the separable projection property and includes McWilliam's result.

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2. Preliminaries. If \( X \) is a topological space, a decomposition \( D \) of \( X \) is a disjoint collection of closed subsets of \( X \) such that \( X = \bigcup \{ A : A \in D \} \). The notation \( X/D \) denotes the set \( D \) with its quotient topology. A set \( A \in D \) is called plural if it contains at least two elements. Also, a set \( A \in D \) is called a limit set if each open set containing \( B \) has nontrivial intersection with a plural set in \( D \sim \{ B \} \). The \( n \)th derived decomposition \( D^{(n)} \) of \( X \) is defined as follows: \( D^{(1)} \) is the decomposition of \( X \) consisting of the plural limit sets in \( D \) and singleton sets. Inductively, if \( D^{(n)} \) is defined, then \( D^{(n+1)} \) is the decomposition of \( X \) consisting of the plural limit sets in \( D^{(n)} \) and singleton sets. If \( D^{(n)} \) contains no plural sets, we write \( D^{(n)} = 0 \). The concept of the \( n \)th derived set is due to R. Arens [3]. A subset \( Y \) of \( X \) is \( D \)-saturated if it is a union of sets in \( D \). Any additional terminology and the basic properties of decompositions used here may be found, for example, in [8].

If \( X \) and \( Y \) are compact Hausdorff spaces and \( \phi \) is a (continuous) map of \( X \) onto \( Y \), then \( \phi^0 \) denotes the isometric isomorphism from \( C(Y) \) into \( C(X) \) that takes \( f \) to \( \phi \circ f \). If \( Y = X/D \) and \( \phi \) is the quotient map of \( D \), then \( \phi^0[C(X/D)] \) is identified with \( C(X/D) \) and consists of all functions in \( C(X) \) which are constant on each set in \( D \). If \( \lambda \) is an ordinal, the topological derivative of order \( \lambda \) of \( X \), denoted \( X^{(\lambda)} \), is defined as follows: \( X^{(0)} = X \), \( X^{(\lambda+1)} = (X^{(\lambda)})' \), and \( X^{(\lambda)} = \bigcap_{\beta < \lambda} X^{(\beta)} \) for \( \lambda \) a limit ordinal, where \( X' \) denotes the derived set of \( X \).

3. Results.

**Lemma.** If \( H \) is a subspace of \( C([0, 1]) \) isometrically isomorphic to \( C(\omega^n k) \), then there is a projection \( \Pi : C([0, 1]) \to H \) with \( \|\Pi\| \leq 2n + 1 \).

**Proof.** Let \( u \) be an isometric isomorphism from \( C(\omega^n k) \) onto \( H \). By a theorem of W. Holsztyński (see [7] or [11]), there is a closed subset \( Q \) of \([0, 1]\), a map \( \phi \) of \( Q \) onto \( \Gamma(\omega^n k) \) and \( \epsilon \) in \( C(Q) \) such that \( |\epsilon(q)| = 1 \) and \( \epsilon(q)(ug)(q) = \phi^0 g(q) \) for all \( q \in Q \) and \( g \in C(\omega^n k) \). Let \( D \) be the upper semicontinuous decomposition \( \{ \phi^{-1}(x) : x \in \Gamma(\omega^n k) \} \) of \( Q \) induced by the closed map \( \phi \). Define \( Q_1 = \phi^{-1}[0, \omega^n] \) and \( Q_i = \phi^{-1}(\omega^n(i-1), \omega^n i) \) for \( 1 \leq i < k \) and let \( D_i \) be the restriction of \( D \) to \( Q_i \). Then \( D_i \) is an u.s.c. decomposition of the compact set \( Q_i \). If \( H_i \) is the u.s.c. decomposition \( D_i^{(1)} \) of \( Q_i \), then \( (\omega^n(i-1), \omega^n i)^{n+1} = \emptyset \) implies \( H_i^{(n)} = 0 \). By Theorem 1.9 in [4], there is a projection \( P_i \) of \( C(Q_i) \) onto \( C(Q_i/H_i) \) with \( \|P_i\| \leq 2n + 1 \). Let \( Y_i \) be the union of the plural sets in \( D_i - H_i \). Each plural set \( S \) in \( D_i - H_i \) is open and closed in \( Q_i \). Let \( \chi(S) \) be a point in \( S \) and for \( f \in C(Q_i) \) define

\[
P_i^* f(x) = P_i f(x), \quad \text{if } x \in Q_i - Y_i
\]
\[
= P_i f(\chi(S)), \quad \text{if } x \in S \subset Y_i.
\]
We show $P_i^*f \in C(Q_i/D_i)$. Since $P_i^*f$ is defined on $Q_i$ and constant on each set in $D_i$, it suffices to show $P_i^*f$ is continuous.

Suppose $x_n \in Q_i$ and $x_n \rightarrow x$. If $x_n \in Q_i - Y_i$ for each $n$, then $x \in Q_i - Y_i$ and $P_i^*f(x_n) \rightarrow P_i^*f(x)$ since $P_i^*f$ agrees with $P_i f$ on $Q_i - Y_i$. Thus, it suffices to suppose each $x_n \in Y_i$. If $x \in Y_i$, then $x \in S$ for some open set $S$ in $D$ and there exists $N>0$ such that $x_n \in S$ for $n \geq N$. Then $P_i^*f(x_n) = P_i^*f(x)$ for $n \geq N$; so $P_i^*f(x_n) \rightarrow P_i^*f(x)$. Therefore, we may assume $x \in Q_i - Y_i$. Let $\delta > 0$ and choose $S \in H_i$ with $x \in S$. Then $P_i f$ is constant on $S$. Thus there exists a $D_i$-saturated neighborhood $U$ of $S$ such that $|P_i f(y) - P_i f(x)| < \delta$ for all $y \in U$. There exists $N > 0$ such that $x_n \in U$ for all $n \geq N$. Therefore, if $x_n \in S_n \subset D_i$, then $S_n \subset U$ and

$$|P_i^*f(x_n) - P_i^*f(x)| = |P_i f(x(S_n)) - P_i f(x)| < \delta$$

for all $n \geq N$. Consequently, $P_i^*f$ is continuous. It follows that $P_i^*$ is a projection of $C(Q_i)$ onto $C(Q_i/D_i)$ with $\|P_i^*\| \leq \|P_i\| \leq 2n + 1$.

Since each $Q_i$ is both open and closed, $C(Q) = C(Q_1) \oplus C(Q_2) \oplus \cdots \oplus C(Q_k)$. For each $f = f_1 + f_2 + \cdots + f_k \in C(Q)$ with $f_j \in C(Q_j)$, let $P f = P_1 f_1 + P_2 f_2 + \cdots + P_k f_k$. Then $P$ is a projection of $C(Q)$ onto $C(Q/D)$ with $\|P\| \leq 2n + 1$. Let $R$ denote the restriction operator from $C(X)$ onto $C(Q)$ and $\epsilon' = 1/\epsilon$. Define $\Pi = u f(\phi^o)^{-1} P T_\epsilon^R$ where $T_\epsilon^R : C(Q) \rightarrow C(Q)$ by $T_\epsilon^R f \equiv \epsilon f$. Clearly $\Pi$ is a continuous linear operator from $C(X)$ into $H$. If $f \in H$, say $f = u g$, then $R f = f Q = \epsilon' \phi^o g$. Therefore,

$$\Pi f = u f(\phi^o)^{-1} P T_\epsilon^R (\epsilon' \phi^o g) = u f(\phi^o)^{-1} P \phi^o g = u g = f.$$

This shows $\Pi$ is a projection. Clearly, $\|\Pi\| \leq 2n + 1$.

**Theorem.** $p_i(C(\omega^n k)) = 2n + 1$.

**Proof.** Let $E$ be a separable Banach space, $S = \Gamma(\omega^n k)$, and $u : C(S) \rightarrow E$ be an isometric embedding. By the Banach-Mazur theorem, we may assume $E$ is a subspace of $C([0, 1])$. By the preceding lemma, there is a projection $\Pi$ of $C([0, 1])$ onto $u[C(S)]$. The restriction $P$ of $\Pi$ to $E$ is a projection of $E$ onto $u[C(S)]$ with $\|P\| \leq 2n + 1$; hence, $C(S) \in P'_{2n+1}$.

The fact that $C(S) \notin P'_\lambda$ for $\lambda < 2n + 1$ is established by Amir in the proof of the theorem in [1]. For completeness sake, we sketch his proof. Let $\Sigma$ be a countable field of subsets of $S$ which contains a basis for the open sets in $S$ and is closed under complements, finite unions, and the closure operation. Denote by $B(S, \Sigma)$ the closed subspace spanned in $m(S)$ by the characteristic functions of the sets of $\Sigma$. Then $B(S, \Sigma)$ is a separable Banach space containing $C(S)$. Since for each $n$-tuple $(k_1, k_2, \cdots, k_n)$ of positive integers, $S$ is $(k_1, k_2, \cdots, k_n)$-$\Sigma$-connected (see [1] for definition) at $\omega^n$, it follows from the lemma in [1] that if $P$ is a projection of $B(S, \Sigma)$ onto $C(S)$, then $\|P\| \geq 2n + 1$. Therefore, $C(S) \notin P'_\lambda$ for $\lambda < 2n + 1$. 

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BIBLIOGRAPHY


12. A. Sobczyk, Projections of the space (m) on its subspace (c₀), Bull. Amer. Math. Soc. 47 (1941), 938–947. MR 3, 205.

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