ON THE BOUNDEDNESS OF PSEUDO-DIFFERENTIAL OPERATORS

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Abstract. In this note, we use the sequence version of Cotlar's lemma and a partition of unity to give a proof of the $L^2$-boundedness of a class of pseudo-differential operators.

Introduction and results. Let $p(x, \xi)$ be a continuous function on $\mathbb{R}^n \times \mathbb{R}^n$. Then, a pseudo-differential operator $P$ with the symbol $p(x, \xi)$ is a linear map of $C^\infty_0(\mathbb{R}^n)$ into $C^0(\mathbb{R}^n)$, defined by

$$P u = \left(\frac{(2\pi)^{1/2}}{i}\right)^n \int e^{ix\xi} p(x, \xi) \hat{u}(\xi) \ d\xi$$

for $u \in C^\infty_0(\mathbb{R}^n)$, where $\hat{u}$ is the Fourier transform of $u$.

In [1] A. Calderón and R. Vaillencourt prove the following

Theorem A. Let $p(x, \xi)$ be a function defined on $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$|1 + \partial_{x_n}^a \cdots (1 + \partial_{\xi_1}^b)(1 + \partial_{\xi_n}^c) p(x, \xi)| \leq C$$

for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$. Then the pseudo-differential operator associated with the symbol $p(x, \xi)$ can be extended to a bounded operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

Using the sequence version of Cotlar's lemma (cf. [2]) and a partition of unity, we can prove the following result analogous to Theorem A.

Theorem. Let $p(x, \xi)$ be a function defined on $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$(1) \quad \int_Q |\partial_x^\alpha \partial_{\xi}^\beta p(x, \xi)| \ dx \leq C, \quad \text{for } 0 \leq \alpha, \beta \leq 3,$$

and all $(x, \xi) \in Q \times \mathbb{R}^n$, where $Q$ is any cube with edges of length two and parallel to the axes. Then the associated operator can be extended to a bounded operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.
Remark. Actually a slight modification of the proof shows that the theorem still holds under the following weaker assumptions on $p(x, \xi)$ instead of (1):

$$\int_Q |\partial_x^\alpha \partial_{\xi}^\beta p(x, \xi)| \leq C$$

and

$$\int_Q |\partial_x^\alpha \partial_{\xi}^\beta p(x, \xi)| \leq C h^\delta$$

for some $\delta > 0$, $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 3$, $0 \leq h \leq 1$ and all $(x, \xi) \in Q \times R^n$.

Proof of Theorem. It suffices to prove the theorem for $n=1$. We use $C$ to denote various constants. Also, we can assume without loss of generality that $p(x, \xi)$ has compact support in $\xi$. Let $f(x)$ be an infinitely differentiable function with $\text{supp} f \subset C(-5/4, 5/4)$ and equal to one for $|x| \leq 1$. We define

$$p_i(x, \xi) = \left( f(x - i) \int_0^\infty F(x - k) \right) p(x, \xi)$$

for all integers $i$. It is easy to show that

$$\int_R |\partial_x^\alpha \partial_{\xi}^\beta p_i(x, \xi)| \leq C$$

for $0 \leq \alpha \leq 2$, $0 \leq \beta \leq 3$ and all $(x, \xi) \in R \times R$.

As the Fourier transform of differentiation is multiplication, we can conclude that

$$(5) |\partial_x^\beta \tilde{p}_i(\eta, \xi)| \leq C/|\eta|^2 \quad \text{for } 0 \leq \beta \leq 3$$

for all $(\eta, \xi) \in R \times R$, where $\tilde{p}_i(\eta, \xi)$ is the Fourier transform of $p_i(x, \xi)$ in the space variable $x$.

As in Kohn-Nirenberg [3], we then have

$$\|P_i^{(\beta)} u\| \leq C \|u\| \quad \text{for } 0 \leq \beta \leq 3,$$

where $P_i^{(\beta)}$ is the operator associated with the symbol $\partial_x^\beta p_i(x, \xi) = p_i^{(\beta)}(x, \xi)$.

Since $p_{2n+1}(x, \xi)$ and $p_{2m+1}(x, \xi)$ have disjoint support in $x$ for $n \neq m$, we have

$$(6) P_{2n+1}^* P_{2m+1} = 0$$

where $P^*$ is the adjoint of $P$; that is

$$(Pu, v) = (u, P^*v) \quad \text{for } u, v \in C_0^\infty(R^1),$$
or

\( P^* v(\xi) = \int e^{-ix\xi} p(x, \xi) v(x) \, dx. \)

Let \( u \) and \( v \) be \( C_0^\infty(R) \) functions and \( I = (P_{2n+1}^*, P_{2m+1}^*). \) Applying Parseval's theorem and using (7) we obtain

\[
I = \int u(x) \overline{v(y)} \, dx \, dy \int e^{i(y-z)\xi} p_{2n+1}(x, \xi) \overline{p_{2m+1}(y, \xi)} \, d\xi
\]

\[
= - \int \frac{u(x) v(y)}{(x-y)^3} \, dx \, dy \int e^{i(y-z)\xi} \frac{\partial^2}{\partial \xi^2} p_{2n+1}(x, \xi) \overline{p_{2m+1}(y, \xi)} \, d\xi
\]

\[
= \sum_{\alpha+\beta=3} C_{\alpha, \beta} \int d\xi \int \hat{u}(n) \, dn \int \hat{v}(\xi) \, d\xi \int H(x, y, \xi, \eta, \zeta) \, dx \, dy
\]

with

\[
H = \frac{e^{i(\xi-\eta)x}}{1 + (\xi - \eta)^2} \frac{e^{i(\xi-\eta)\xi}}{1 + (\xi - \eta)^2} \left(1 - \frac{\partial^2}{\partial x^2}\right)
\]

\[
\cdot \left(1 - \frac{\partial^2}{\partial y^2}\right) p_{2m+1}(y, \xi) p_{2n+1}(x, \xi).
\]

As \( 1/(x-y)^k \leq C/(m-n)^{k+1} \) for \( k \geq 3, \) \( x \in \text{supp} p_{2n+1}(x, \xi) \) and \( y \in \text{supp} p_{2m+1}(y, \xi) \) with \( n \neq m, \) we can conclude from (8) that

\[
|I| \leq \left[ \int d\xi \int \frac{\hat{u}(n)}{1 + (\xi - n)^2} \, dn \int \frac{|\hat{v}(\xi)|}{1 + (\xi - \xi)^2} \, d\xi \right] \frac{c}{(m-n)^3 + 1}
\]

\[
\leq \frac{c}{(m-n)^3 + 1} \|u\| \|v\|.
\]

In virtue of (6) and (9), we can apply Cotlar's lemma to deduce the fact that

\( \sum_{n=1}^{\infty} P_{2n+1} \leq C. \)

Similarly, we can prove

\( \sum_{n=0}^{\infty} P_{2n} \leq C. \)

(10) and (11) imply that \( P \) can be extended to a bounded operator from \( L^2 \) to \( L^2. \)

**REFERENCES**


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