BOREL'S FIXED POINT THEOREM FOR KAELER MANIFOLDS AND AN APPLICATION

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Abstract. A short proof of a generalization of the Borel fixed point theorem to the case of Kaehler manifolds is given and, as an application, a short proof of Wang's theorem that compact simply connected homogeneous manifolds are projective and of the form \( G/P \), where \( G \) is a complex semisimple Lie group and \( P \) is a parabolic subgroup.

I will give a short proof of a generalization of the Borel fixed point theorem to the case of Kaehler manifolds and, as an application, give a short proof of Wang's theorem that compact simply connected homogeneous Kaehler manifolds are projective and of the form \( G/P \), where \( G \) is a complex semisimple Lie group and \( P \) is a parabolic subgroup.

I would like to thank Phillip Griffiths who suggested trying to find a short proof of Wang's theorem. I would also like to thank Professor Yozo Matsushima for his comments.

Proposition I. Let \( X \) be a compact Kaehler manifold with \( H^1(X, \mathbb{C}) = 0 \), and let \( S \) be a solvable connected complex Lie group acting holomorphically on \( X \). Let \( Y \) be a subvariety of \( X \) invariant under \( S \). Then \( S \) has a fixed point on \( Y \) and the fixed points form a subvariety.

Proof. First assume \( Y \) is a manifold, \( S \) is one dimensional and has no fixed points on \( Y \). Associated to \( S \) we have a holomorphic tangent field on \( X \) and by invariance of \( Y \) under \( S \), also on \( Y \); call it \( A \). By assumption \( A \) has no zeroes on \( Y \).

We have short exact sequences where \( \mathcal{O}_Y, \mathcal{O}_X \) are the holomorphic structure sheaves and \( \Omega^1_X, \Omega^1_Y \) are the holomorphic one forms and \( M \) is a subsheaf of \( \mathcal{O}_X \). \( A \) and \( A|_Y \) as sections of the dual sheaves of \( \Omega^1_X \) and \( \Omega^1_Y \).

Received by the editors December 7, 1972.


Key words and phrases. Transcendental algebraic geometry and Hodge theory, homogeneous manifolds, automorphism groups of complex manifolds, Kaehler manifolds.
\( \Omega^1_Y \) respectively give rise to natural maps denoted by the same letters. \( r \) stands for the restriction map from \( X \) to \( Y \).

\[
\begin{array}{c}
0 \rightarrow F^1 \overset{A}{\rightarrow} \Omega^1_X \overset{\mathcal{O}_X}{\rightarrow} 0 \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
0 \rightarrow F^2 \overset{A^1_Y}{\rightarrow} \Omega^1_Y \overset{\mathcal{O}_Y}{\rightarrow} 0
\end{array}
\]

\( A \) having no zeroes is equivalent to \( \Omega^1_Y \overset{A^1_Y}{\rightarrow} \mathcal{O}_Y \) being a surjection, and \( F^2 \) being locally free of rank \( n-1 \).

Passing to cohomology we have the commutative diagram:

\[
\begin{array}{ccc}
H^1(X, \Omega^1_X) & \overset{f^1_X}{\longrightarrow} & H^1(X, \mathcal{O}_X) \\
\downarrow & & \downarrow \\
H^1(Y, \Omega^1_Y) & \overset{f^1_Y}{\longrightarrow} & H^1(Y, \mathcal{O}_Y)
\end{array}
\]

The Kaehler form \( \omega \) on \( X \) restricts to a Kaehler form \( \omega|_Y \) on \( Y \). Since \( H^1(X, \mathcal{O}_X) \) is a subgroup of \( H^1(X, \mathcal{O}_X) \) on a Kaehler manifold, it equals zero. Thus the image of \( \omega|_Y \) in \( H^1(Y, \mathcal{O}_Y) \) is 0 and thus the Kaehler form of \( Y \) is an element of \( H^1(Y, F^2) \). By the Dolbeault isomorphism we can represent \( \omega|_Y \) by a \( F^2 \) valued 0, 1 form. I observe that the \( n \)th exterior power of \( \omega|_Y \), where the dimension of \( Y = n \), would be zero, since the fibre dimension of \( F^2 = n-1 \). On the other hand the \( n \)th power of \( \omega|_Y \) is a nontrivial element of \( H^{2n}(Y, \mathcal{O}_Y) \), a volume form, which gives a contradiction.

We now remove the assumption that \( Y \) is nonsingular. Simply note that \( Y \) has a finite filtration by singular varieties

\[
Y = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_m,
\]

where \( Y_s \) is the singular set of \( Y_{s-1} \). This filtration is respected by \( S \) and the last element of the filtration is a manifold which we can take as \( Y \).

The fixed point set is the zero set of \( A \) and so is a subvariety.

Now let \( S \) be of dimension \( n \) and assume the proposition is true for \( n-1 \). Since \( S \) is solvable it has a normal subgroup \( S' \) of dimension \( n-1 \). By assumption \( S' \) has a nontrivial fixed point variety \( Y' \) in \( Y \). Note that \( S \) leaves this new \( Y' \) invariant. To see this, take element \( s \) of \( S \), and an element \( y \) of \( Y' \) and note that \( S'(sy) = s(S'y) = sy \), so \( sy \) is a fixed point of \( S' \) in \( Y \), that is, an element of \( Y' \).

Pick an \( A \) belonging to the complex Lie algebra of \( S \) and not belonging
to the Lie subalgebra of $S'$. $A$ gives rise to a one parameter subgroup of $T$ the universal cover of $S$, and hence to a tangent vector field on $Y'$ and the fixed point set of $S$ in $Y$ is the fixed point set of $T$ in $Y'$, and the first part of the proof applies. Q.E.D.

**Proposition II.** Let $X$ be a compact homogeneous Kaehler manifold with $H^1(X, C)=0$. Then $X$ is a projective manifold of the form $G/P$ where $G$, the connected component of the identity of $X$'s complex Lie group of biholomorphic transformations is semisimple, and $P$ is parabolic, that is, contains a maximal solvable connected subgroup.

**Proof.** If $G$ were not semisimple it would have a solvable radical $N$. Let $x \in X$ be a fixed point of $N$ which exists by Proposition I. For all $g \in G$ we have $N(gx)=g(Nx)=gx$. But $G$ is transitive and so $N$ leaves every point fixed and is thus the identity.

Let $B$ be a maximal solvable connected subgroup of $G$. $B$ has a fixed point $x$ and so $B$ belongs to the stabilizer of $x$, which we call $P$. Thus $x$ is of the form $G/P$ where $P$ is parabolic. Q.E.D.

**Remarks.** It is easy to give simply connected complex manifolds where the Borel fixed point theorem is false, e.g. the Calabi-Eckmann manifolds. The correct setting is probably on an appropriate generalization of Kaehler manifolds that one might call pseudo-Kaehler manifolds. These would be complex compact manifolds with a closed $C^\infty$ 1, 1 form which is positive definite on a Zariski open set. Hodge theory should go through for these manifolds, and then the above proof will be applicable.

The general structure of a compact Kaehlerian homogeneous space is given in [1].

The proof of the fixed point theorem above, looked at in the context of the Albanese map, actually proves a strong converse.

**Proposition.** Let $S$ be a complex connected solvable Lie group acting holomorphically on a compact Kaehler manifold $X$. $S$ has a fixed point on any subvariety, including $X$, that $S$ leaves invariant if and only if the complex Lie algebra of holomorphic vector fields on $X$ associated to $S$ is annihilated by every holomorphic one form.

If $S$ has a fixed point, then the fixed point subvariety surjects onto the image of $X$ in its Albanese variety under the Albanese map.

In the case that there are no holomorphic one forms, that is $H^1(X, C)=0$, the above result reduces to the theorem in the paper.

A corollary of the above proposition is that a compact Kaehler manifold that possesses a nowhere vanishing holomorphic vector field $A$, also possesses a holomorphic one form $a$ such that $a(A) \neq 0$.

This proof and other related results will be presented in a future article.
BIBLIOGRAPHY


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