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AN EXAMPLE CONCERNING NORMED SUBSPACES OF LOCALLY CONVEX SPACES

ROBERT H. LOHMAN

ABSTRACT. An example is given of a nonmetrizable, nonbarrelled locally convex space that is the algebraic direct sum of two subspaces, each of which is a Banach space in the relative topology.

Rosenthal has shown in [3] that if X is a normed linear space and Y, Z are subspaces of X such that Y and Z are totally incomparable Banach spaces, then $Y+Z$ is a Banach space. Rosenthal's result was generalized by Diestel and Lohman in [1, Theorem 4], where it was shown that the latter theorem remains true if X is assumed to be a locally convex linear topological space (henceforth an LCLT space). In view of these results, it is natural to ask whether it is always the case that the sum of two normable subspaces of a LCLT space is normable. The purpose of this paper is to give an example of a nonmetrizable, nonbarrelled LCLT space that is the algebraic direct sum of two subspaces, each of which is a Banach space in the relative topology.

We let $m(\Gamma)$ denote the set of bounded, real-valued functions on the set Γ with the supremum norm. If Γ is countable, $m(\Gamma)$ is denoted by m.

LEMMA. *Let F and G be closed linear subspaces of a Banach space H such that $F\cap G=\{0\}$ and $F+G$ is not closed in H. Then there exists a discontinuous linear functional f on $F+G$ such that $f|_F$ is continuous and $f|_G=0$.*

PROOF. Consider the projection $P:F+G\to F$ defined by $P(x+y)=x$ for all $x\in F$, $y\in G$. Since $F+G$ is not closed in H, P is not continuous. Therefore, P is not weakly continuous. It follows that there exists a continuous linear functional g on F such that gP is not continuous. If we let $f=gP$, then $f|_F=g$ is continuous and $f|_G=0$.

EXAMPLE. Let Γ be an uncountable set and let $\{\Gamma_\alpha:\alpha\in\Sigma\}$ be a partition of Γ into an uncountable disjoint family of infinitely countable sets. Let $F=c_0$ and let G be a closed subspace of m such that $F\cap G=\{0\}$ and $F+G$ is not closed in m. For example, G can be a quasi-complement

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of c_0 in m (see [4]). Let

$$Y = \{y \in m(\Gamma): y|_{\Gamma_\alpha} \in F \text{ for all } \alpha \in \Sigma\},$$
$$Z = \{z \in m(\Gamma): z|_{\Gamma_\alpha} \in G \text{ for all } \alpha \in \Sigma\}.$$

Then Y and Z are closed in $m(\Gamma)$, $Y \cap Z = \{0\}$, and $Y+Z$ is not closed in $m(\Gamma)$.

By the lemma, there exists a discontinuous linear functional f on $F+G$ such that $f|_F$ is continuous and $f|_G = 0$. For each $\alpha \in \Sigma$ let

$$G_\alpha = \{x \in Y + Z: x(\gamma) = 0 \text{ if } \gamma \notin \Gamma_\alpha\},$$
$$H_\alpha = \{x \in Y + Z: x(\gamma) = 0 \text{ if } \gamma \in \Gamma_\alpha\},$$

and define the linear functional f_α on $Y+Z$ by $f_\alpha|_{G_\alpha} = f$, $f_\alpha|_{H_\alpha} = 0$. Then $f_\alpha|_Y$ is continuous, $f_\alpha|_Z = 0$, and $f_\alpha|_{G_\alpha}$ is not continuous for each $\alpha \in \Sigma$. If $\|\cdot\|$ denotes the norm on $m(\Gamma)$, then the family of sets of the form

$$V_{\varepsilon,\sigma} = \varepsilon\{x \in Y + Z: \|x\| \leqq 1,\ |f_\alpha(x)| \leqq 1 \text{ for all } \alpha \in \sigma\},$$

where $\varepsilon > 0$ and σ is an arbitrary subset of Σ, constitutes a fundamental system of neighborhoods of 0 for a locally convex topology τ on $X = Y+Z$. Because $f_\alpha|_Y$ and $f_\alpha|_Z$ are norm-continuous, the relativization of τ to Y and Z is the norm topology, so that Y and Z are Banach spaces in the relative topology of τ.

If we assume that τ is metrizable, then there exist positive numbers $\varepsilon_n > 0$ and finite subsets $\sigma_n \subset \Sigma$ such that the sequence $V_{\varepsilon_n,\sigma_n}$, $n = 1, 2, \cdots$, constitutes a fundamental system of τ-neighborhoods of 0. Since Σ is uncountable, there exists an index $\beta \in \Sigma \setminus \bigcup_{n=1}^{\infty} \sigma_n$. Each f_α is τ-continuous, so that there exists a positive integer m such that $f_\beta(V_{\varepsilon_m,\sigma_m})$ is a bounded set. It follows that f_β is norm-continuous on $\bigcap_{\alpha \in \sigma_m} \ker f_\alpha$. Since the latter set contains G_β, f_β must be norm-continuous on G_β. This contradiction shows that τ is not a metrizable vector topology. It follows from the generalized closed graph theorem [2, p. 301] that E cannot be barrelled when it has the topology τ.

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Department of Mathematics, Kent State University, Kent, Ohio 44242