COVERING DIMENSION IN FINITE-DIMENSIONAL METRIC SPACES

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Abstract. Let $P:2^V \to 2^V$ be a structure in a topological space $V$ such that $P(\emptyset) = \emptyset$, $P(\{x\}) = \{x\}$ if $x \in V$, and $P(Z)$ is closed if $Z \subseteq V$. If $G$ is a covering of $V$, let $G_x = \{X \subseteq G : x \in X\}$. If $X$ is a set and $Y$ is a set, let $|X|$ denote the cardinal number of $X$ and $X - Y = \{x \in X : x \notin Y\}$. Let $n$ be an integer such that $n \geq -1$. $\dim P V$ is defined as follows: $\dim P V = -1$ if $V = \emptyset$. If $V \neq \emptyset$, then $\dim P V = n$ if (1) for each finite open covering $G$ of $V$, there is an open refinement $H$ of $G$ such that $|H_x| \leq n + 1$ if $x \in V$; and (2) there is a finite open covering $G$ of $V$ such that if $H$ is an open refinement of $G$, then $|H_x| \geq n + 1$ for some $x \in V$. We say that $P$ has property (*) if for each nonempty open $Y \subseteq V$ and each $X \subseteq V$ such that $P(X) \subseteq V$ and $x \notin P(X - \{x\})$ whenever $x \in Y$ and each $x \in \{V - P(X)\} \cap P(X \cup \{x\}) \neq \emptyset$. Theorem. If $V$ is a metric space, $P$ has property (*), $B \subseteq V$, $B$ is finite, $P(B) = V$ and $x \notin P(B - \{x\})$ if $x \in B$, then $\dim P V = |B| - 1$.

1. Introduction. It is known [5, pp. 9, 93–99] that the covering dimension of each finite-dimensional Euclidean space $E^n$ is $n$, the usual dimension. The purpose of this paper is to present a short proof of this simply stated fact.

It is crucial that each finite-dimensional Euclidean space is a topological space $V$ in which there is a structure $P:2^V \to 2^V$ [4, p. 317] such that $P$ is a closure structure having the exchange property ([2], [3], and [4]), $P(\emptyset) = \emptyset$, $P(\{x\}) = \{x\}$ for each $x \in V$, and $P(Z)$ is closed for each $Z \subseteq V$. Indeed, if $V = E^n$, then the linear variety structure in $V$ will suffice as $P$, that is, if $X \subseteq V$, then $P(X)$ is the collection of all finite linear combinations of elements of $X$ with coefficients summing to 1.

Consider a structure $P$ in a set $V$ and a subset $X$ of $V$. By definition, $X$ is $P$-independent ([2] and [3]) if $x \notin P(X - \{x\})$ for each $x \in X$; $X$ is a $P$-basis of $V$ if $X$ is $P$-independent and $P(X) = V$. By definition, the $P$-dimension of $V$, $P$-dim $V$, exists if any two $P$-bases of $V$ have the same cardinal number. If $P$-dim $V$ exists, then $P$-dim $V$ is the cardinal number.
of a P-basis of V. It is known ([2] and [3]) that if P is a closure structure having the exchange property and V has a finite P-basis, then P-dim V exists.

If G is a covering of a set V and x ∈ V, then the symbol G_x will denote \{X ∈ G: x ∈ X\}. If X is a set and Y is a set, the symbol X − Y will denote \{x ∈ X: x ∉ Y\}, and the symbol |X| will denote the cardinal number of X. Throughout the remainder of this paper it is assumed that V is a topological space and P is a structure in V such that P(∅) = ∅, P(\{x\}) = \{x\} for each x ∈ V and P(Z) is closed for each Z ⊆ V.

The covering dimension of V relative to P, dim_P V, is defined as follows: dim_P V = −1 if V = ∅. If V ≠ ∅ and n is a cardinal number, then dim_P V = n if (1) and (2) are true: (1) For each finite open covering G of V, there is an open refinement H of G \[an open covering of V such that if X ∈ H, then X ⊆ Y for some Y ∈ G]\ such that |H_x| ≤ n + 1 for each x ∈ V, and (2) There is a finite open covering G of V such that if H is an open refinement of G, then |H_x| ≥ n + 1 for some x ∈ V.

We say that P has property (⋆) if for each nonempty open subset Y of V and each P-independent subset X of V such that X is not a P-basis of V and each x ∈ V − P(X), Y − P(X) contains an element of P(X ∪ {x}).

It is shown (Theorem 1) that if G is a finite open covering of V and B is a P-basis of V, then there is an open refinement H, of G such that \(|(H)_{B,x}| ≤ |B|\) for each x ∈ V; and (Theorem 2) that if V is a metric space and P has property (⋆) while B is a finite P-basis of V, then there is a finite open covering G _{H} of V such that if H is an open refinement of G _{B}, then \(|H_x| ≥ |B|\) for some x ∈ V. It follows (Theorem 3) that if V is a metric space and P has property (⋆) while B is a finite P-basis of V, then dim_P V = |B| − 1.

2. Main results. If V is a metric space, then the following notation will be used: If r is a positive real number and A' is a nonempty subset of V, then the symbol XR shall denote \{x ∈ V: d(x, X) < r\}, where d is the metric on V. The term “poset” [1, p. 1] will be used to refer to a pair (W, R) such that W is a set and R is a partial order relation on W.

**Theorem 1.** If G is a finite open covering of V and B is a P-basis of V, then there is an open refinement H, of G such that \(|(H)_{B,x}| ≤ |B|\) for each x ∈ V.

**Proof.** Assume that G is a finite open covering of V, and that B is a P-basis of V. Since P(∅) = ∅ and P(\{x\}) = \{x\} for each x ∈ V, it follows that if B = ∅ or B = \{x\} for some x ∈ V, then V = P(B) = B, so that G = \{V\}. Hence, if B = ∅ or B = \{x\} for some x ∈ V, then let H _{B} = G. Consider the case that |B| > 1. Let b ∈ B. Using Hausdorff’s maximal principle [1,
p. 192], extend the chain \( \{ \emptyset \} \) of the poset \( (2^{B-(b)}, \subseteq) \) to a maximal chain \( K \) of \( (2^{B-(b)}, \subseteq) \). Since \( G \) is finite, then \( \bigcap G_x \) is open for each \( x \in V \) and each element of the poset \( \{ (G_x : x \in V), \subseteq \} = \{ (G_x : x \in V) \) is preceded by some minimal element of \( \{ G_x : x \in V \} \). Since \( P \) is monotone and \( B \) is \( P \)-independent, it follows that each subset of \( B \) is \( P \)-independent. Hence, since \( \emptyset \in K \) and \( P(\emptyset) = \emptyset \), it follows that the collection \( H \) of all \( (\bigcap G_u) \cap [V-P(X)] \) such that \( G_x \) is a minimal element of \( \{ G_y : y \in V \} \) and \( X \in K \) is an open refinement of \( G \). Assume that \( x \in V \). Choose a minimal element \( G_x \) of \( \{ G_y : y \in V \} \) such that \( G_x \subseteq G_e \). Then \( \bigcap G_x \subseteq \bigcap G_e \) while \( x \in \bigcap G_x \). It follows that the elements of \( H_x \) are among the sets \( \{ (\bigcap G_u) \cap [V-P(X)] \) such that \( X \in K \) and \( G_y \) is a minimal element of \( \{ G_z : z \in V \} \) while \( |\{ V-P(X) : X \in K \}| = |B| \). Therefore, \( |H_x| \leq |B| \). Let \( H_B = H \). The proof is complete.

**Theorem 2.** If \( V \) is a metric space and \( P \) has property (*) while \( B \) is a finite \( P \)-basis of \( V \), then there is a finite \( P \)-basis \( G_B \) of \( V \) such that if \( H \) is an open refinement of \( G_B \), then \( |H_x| \geq |B| \) for some \( x \in V \).

**Proof.** Assume that \( V \) is a metric space such that \( P \) has property (*) while \( B \) is a finite \( P \)-basis of \( V \). If \( B = \emptyset \) or \( B = \{ x \} \) for some \( x \in V \), then let \( G_B = \{ V \} \). Consider the case that \( |B| > 1 \). Let \( n \) be a positive integer such that \( |B| = n+1 \). Let \( B \) consist of exactly \( n+1 \) elements \( x_i \) of \( V \), with \( 1 \leq i \leq n+1 \). Let \( B_{n+1} = B \). If \( 0 \leq k \leq n \), then let \( B_{n-k} = X_{(n-k)+1} - \{ x_{n-k} \} \).

Let \( r \) be a positive real number. Let \( G \) consist of precisely the following sets: \( P(X_k)_r \), with \( 1 \leq k \leq n \) and \( X_0 = \emptyset \). Since \( P(Z) \) is closed for each \( Z \subseteq V \), it follows that \( G \) is a finite collection of open subsets of \( V \). Consider any element \( x \) of \( V \). If \( x \in P(X_1) \), then \( x \in [P(X_1)_r-P(X_{k-1})] \subseteq G \). If \( x \notin P(X_1) \), let \( m \) be the largest positive integer such that \( x \in P(X_m) \), so that \( x \in [P(X_m+1)_r-P(X_m)] \subseteq G \). It follows that \( G \) is a finite open covering of \( V \). Assume that \( H \) is an open refinement of \( G \). If \( 1 \leq k \leq n+1 \), let \( H_k \) be the collection of all \( X \subseteq H \) such that \( x \in X \) for some \( x \in [P(X_k)_r-P(X_{k-1})] \). Since \( x \in [P(X_1)_r-P(X_0)] \), then \( H_1 \) contains an element \( Y_1 \). It follows from (*) that \( Y_1 \in P(X_1)_r \) contains an element \( y_1 \) of \( P(X_1) \), so that \( y_1 \in [P(X_2)_r-P(X_1)] \). Hence \( y_1 \in Y_2 \) for some \( y_2 \in H_2 \) such that \( y_2 \notin H_1. \) If \( n \geq 2 \), then it follows from (*) that \( (Y_1 \cap Y_2)_r-P(X_2) \) contains an element \( y_2 \) of \( P(X_2) \), so that \( y_2 \in [P(X_3)_r-P(X_2)] \) and \( y_2 \notin [P(X_3)_r-P(X_{i-1})] \) if \( 1 \leq i \leq 2 \). It follows by induction that there is a collection \( \{ Y_i : 1 \leq i \leq n \} \) of elements of \( H \) such that \( 1 \leq k \leq n \) and \( \{ Y_i : 1 \leq i \leq k \} \) contains an element \( y_k \) of \( P(X_k)_r \) while \( y_k \notin H_i \) if \( 1 \leq i < k \). It follows that \( y_n \in [P(X_{n+1})_r-P(X_n)] \) and \( y_n \notin [P(X_1)_r-P(X_1)] \) if \( 1 \leq i \leq n \). Let \( x = y_n \). Then \( x \in Y_{n+1} \) for some \( y_{n+1} \in H_{n+1} \) such that \( y_{n+1} \notin H_i \) if \( 1 \leq i \leq n \). It follows that \( \{ Y_i : 1 \leq i \leq n+1 \} \subseteq H_n \), so that \( |H_n| \geq n+1 \). Therefore, \( |H_n| \geq |B| \) for some \( y \in V \). The proof is complete.
Theorem 3. If $V$ is a metric space and $P$ has property $(\ast)$ while $B$ is a finite $P$-basis of $V$, then $\dim_P V = |B| - 1$.

Proof. Suppose that $V$ is a metric space such that $P$ has property $(\ast)$ while $B$ is a finite $P$-basis of $V$. It follows from Theorem 1 that if $G$ is a finite open covering of $V$, then there is an open refinement $H_B$ of $G$ such that $|(H_B)_x| \leq |B|$ for each $x \in V$. Application of Theorem 2 yields a finite open covering $G_B$ of $V$ such that if $H$ is an open refinement of $G_B$, then $|H_x| \geq |B|$ for some $x \in V$. Therefore, $\dim_P V = |B| - 1$. The proof is complete.

Corollary. If $V$ is a metric space and $P$ has property $(\ast)$ while $V$ has a finite $P$-basis and $P$-dim $V$ exists, then $\dim_P V = [P \text{-dim } V] - 1$.

The linear variety structure $Q$ in $E^n$ is a closure structure having the exchange property and property $(\ast)$, $Q$-dim $E^n$ exists and $E^n$ has a finite $Q$-basis of exactly $n+1$ elements. Therefore, $\dim_Q E^n = n$.

References


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