A CLASS OF PURE SUBGROUPS OF COMPLETELY DECOMPOSABLE ABELIAN GROUPS

DAVID M. ARNOLD

Abstract. Direct sum decompositions of the class of pure subgroups of finite rank completely decomposable torsion free abelian groups with typesets of cardinality at most 4 are considered. In certain cases, the indecomposable groups are classified, resulting in new proofs of several theorems by T. B. Cruddis.

The class $\mathcal{R}$ of pure subgroups of completely decomposable torsion free abelian groups of finite rank is considered by Butler [2]. Many of the examples of pathological direct sum decompositions of finite rank torsion free abelian groups occur in the class $\mathcal{R}$. On the other hand, homogeneous $\mathcal{R}$-groups are completely decomposable and if $A$ is an $\mathcal{R}$-group, then typeset($A$) is finite and closed under intersection of types.

This note considers $\mathcal{R}$-groups with typesets of cardinality at most 4. Motivation is provided by the curious results of Cruddis [3]. An easier and more illuminating proof of these results is a consequence of the following theorems.

Let $\Lambda = \{\tau_0, \tau_1, \tau_2, \tau_3\}$ be a set of types with $\tau_0 = \tau_1 \cap \tau_2 \cap \tau_3$; let $\mathcal{R}_\Lambda$ be the class of $\mathcal{R}$-groups with typeset $\subseteq \Lambda$; let $\sum_\Lambda$ be the sum of types in $\Lambda$; and let $\sum_A$ be the sum of types in typeset($A$). Define $\tau_\infty$ to be the type represented by $\langle \infty \rangle$ and, for $p$ a prime, let $\tau_p$ be the type represented by $(n_0)$ where $n_p = 0$ and $n_q = \infty$ if $p \neq q$.

**Theorem 1.** Suppose that $A$ is an $\mathcal{R}_\Lambda$-group and that typeset($A$) is a proper subset of $\Lambda$.

(a) If $A$ is indecomposable, then rank $A \leq 2$.

(b) If $\sum_\Lambda = \tau_\infty$, then $A$ is completely decomposable.

(c) If $\sum_\Lambda = \tau_p$, for some prime $p$, then $A$ has a unique decomposition (up to isomorphism) into indecomposable summands.

**Theorem 2.** Suppose that $A$ is an $\mathcal{R}_\Lambda$-group with typeset($A$) $= \Lambda$ and that $\sum_\Lambda = \tau_\infty$.

(a) If $A$ is indecomposable, then rank $A \leq 3$. 

Received by the editors February 7, 1973.


Key words and phrases. Torsion free abelian groups of finite rank, completely decomposable groups.

© American Mathematical Society 1973

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

37
(b) There is a completely decomposable $\mathcal{R}_A$-group $H$ such that $A \oplus H = G_1 \oplus \cdots \oplus G_k$ and $\text{rank}(G_i) \leq 2$ for $1 \leq i \leq k$.

Corollary 1.2 includes an explicit characterization of all indecomposable $\mathcal{R}_A$-groups, where $\sum_A = \tau_\infty$.

Let $\pi = \{p, q, r\}$ be a set of distinct primes, let $\mathcal{G}_3$ be the class of $\mathcal{R}$-groups divisible by all primes not in $\pi$, and let $\Lambda = \{\tau_0, \tau_1, \tau_2, \tau_3\}$ where $\tau_0 = \tau_p \cap \tau_q \cap \tau_r$, $\tau_1 = \tau_p \cap \tau_q$, $\tau_2 = \tau_p \cap \tau_r$, and $\tau_3 = \tau_q \cap \tau_r$.

**Corollary 3 (Cruddis).** Let $A$ be a $\mathcal{G}_3$-group.

(a) $A = H \oplus G$ where $H$ is completely decomposable and $G$ is an $\mathcal{R}_A$-group.

(b) Any two decompositions of $A$ into indecomposable summands with rank $\leq 2$ are equivalent.

(c) $K_0(\mathcal{G}_3)$ is a free abelian group with $S = \{[A] | A$ is indecomposable and rank $A \leq 2\}$ as a basis ($K_0(\mathcal{G}_3)$ is the Grothendieck group of $\mathcal{G}_3$, modulo direct sums, and $[A]$ is the element of $K_0(\mathcal{G}_3)$ determined by $A$).

0. Preliminaries and notation. The basic reference for the elementary notions of torsion free abelian groups is Fuchs [4].

A height sequence is a sequence $(n_p)$ of extended integers indexed by the set of primes of $\mathbb{Z}$, such that $0 \leq n_p \leq \infty$ for each prime $p$. Two height sequences $(n_p)$ and $(m_p)$ are equivalent if $n_p = m_p$ for all but a finite number of primes $p$ with $n_p < \infty$ and $m_p < \infty$. A type is an equivalence class of height sequences. The natural ordering of height sequences induces an ordering on the set of all types (we write $\tau_1 \cap \tau_2$ for $\min\{\tau_1, \tau_2\}$) and addition of height sequences (with the convention that $\infty + k = \infty$) induces addition of types.

Assume that $A$ is a torsion free abelian group, that $0 \neq a \in A$, and that $p$ is a prime. Then $h_p(a)$ is the largest integer $n$ with $a \in p^nA \setminus p^{n+1}A$ and $\infty$ if no such $n$ exists. Define $h_A(a)$ to be the height sequence $(h_p(a))$; the type $T_A(a)$ of $a$ in $A$ is the equivalent class of $h_p(a)$. Note that

$$T_A(a_1 + \cdots + a_n) \geq T_A(a_1) \cap \cdots \cap T_A(a_n)$$

and if $A = A_1 \oplus \cdots \oplus A_n$ with $a_i \in A_i$, then

$$T_A(a_1 + \cdots + a_n) = T_A(a_1) \cap \cdots \cap T_A(a_n).$$

Define typeset($A$) to be $\{T_A(a) | 0 \neq a \in A\}$; $A$ is homogeneous if typeset($A$) is a singleton. If rank $A = 1$, then $A$ is homogeneous. Call $A$ completely decomposable if $A$ is the direct sum of rank 1 groups. Pure subgroups of finite rank homogeneous completely decomposable groups are summands.

If $\tau$ is a type, then $A(\tau) = \{x \in A | T_A(x) \geq \tau\}$ is a pure fully invariant subgroup of $A$. If $A = B \oplus C$, then $A(\tau) = B(\tau) \oplus C(\tau)$ and $A / A(\tau)$ is isomorphic to $B / B(\tau) \oplus C / C(\tau)$. Two abelian groups $A$ and $B$ are quasi-isomorphic if there are isomorphic subgroups $A'$ and $B'$ of $A$ and $B$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
respectively, such that \( A/A' \) and \( B/B' \) are bounded. The group \( A \) is *strongly indecomposable* if \( A \) quasi-isomorphic to \( B \oplus C \) implies that \( B = 0 \) or \( C = 0 \).

1. \( R_A \)-groups.

**Lemma 1.1 (Butler [2]).** Let \( A \) be an \( R \)-group with typeset(\( A \)) where

\[
\tau_0 = \tau_1 \cap \cdots \cap \tau_n.
\]

(a) If typeset(\( A \)) is well ordered, then \( A \) is completely decomposable.

(b) If \( A \) is indecomposable, then \( A/(A(\tau_1) + \cdots + A(\tau_n)) \) is a finite group. Moreover, if \( \tau_0 = \tau_i \cap \tau_j \) for \( 1 \leq i \neq j \leq n \), then \( A(\tau_i) \cap A(\tau_j) = 0 \) whenever \( i \neq j \).

**Proof of Theorem 1.** In view of Lemma 1.1(a), we may assume that typeset(\( A \)) = \( \{\tau_0, \tau_1, \tau_2\} \) where \( \tau_0 = \tau_1 \cap \tau_2 \), \( \tau_0 \neq \tau_1 \), and \( \tau_0 \neq \tau_2 \). If \( A \) is indecomposable, then \( A/(A(\tau_1) + A(\tau_2)) = C_1 \oplus \cdots \oplus C_k \), where each \( C_i \) is a cyclic group of order \( m_i \) and \( m_i \) divides \( m_{i+1} \). Let \( A = A(\tau_1) + A(\tau_2) + Zx_1 + \cdots + Zx_k \), where \( x_i + A(\tau_1) + A(\tau_2) \) is a generator of \( C_i \) and \( m_i x_i = a_{i1} + a_{i2} \) for some \( a_{ij} \in A(\tau_j) \) (\( Zx \) is the subgroup of \( A \) generated by \( x \)).

(a) Assume \( A \) is indecomposable with rank \( \geq 1 \) and let \( G_i \) be the pure subgroup of \( A \) generated by \( \{a_{ij}, a_{i2}\} \) for \( 1 \leq i \leq k \). Note that \( a_{i1} \) and \( a_{i2} \) are both nonzero since \( A(\tau_j) \) is a pure subgroup of \( A \) and \( \tau_0 \neq 0 \). Thus \( rank(G_i) = 2 \).

Let \( S = \{a_{11}, \ldots, a_{1k}, a_{21}, \ldots, a_{2k}\} \) and suppose that \( l_1 a_{11} + \cdots + l_k a_{1k} + n_1 a_{21} + \cdots + n_k a_{2k} = 0 \), for some \( i, n_i \in \mathbb{Z} \). Since \( A(\tau_1) \cap A(\tau_2) = 0 \), \( \sum l_i a_{1i} = 0 \) and \( \sum l_i m_i x_i \in A(\tau_2) \). Thus \( \sum r_i x_i \in A(\tau_2) \), where \( r_i = (l_im_i)/d \) and \( d \) is the greatest common divisor of \( \{l_1 m_1, \ldots, l_k m_k\} \). Consequently, \( \sum (r_i x_i + A(\tau_1) + A(\tau_2)) = 0 \in C_1 \oplus \cdots \oplus C_k \) and \( m_i \) divides \( r_i \), a contradiction unless \( l_1 = l_2 = \cdots = l_k = 0 \). Similarly, \( n_i = 0 \) for \( 1 \leq i \leq k \) and \( S \) is a \( Z \)-independent subset of \( A \).

One can easily verify that \( G = G_1 \oplus \cdots \oplus G_k \leq A \) and that \( A(\tau_j) = (G \cap A(\tau_j)) \oplus A_j' \) for some \( A_j' \), where \( j = 1, 2 \) (\( A(\tau_j) \) is a homogeneous completely decomposable group of type \( \tau_j \)). But \( A(\tau_j) \oplus A(\tau_2) \subseteq G \oplus A_j' \oplus A_2' \subseteq A \) and \( x_1, \ldots, x_k \in G \) so \( A = G_1 \oplus \cdots \oplus G_k \oplus A_1 \oplus A_2 \). Since \( A \) is indecomposable, \( A = G_k, k = 1 \), and \( A/(A(\tau_1) + A(\tau_2)) \) is cyclic.

(b) Assume that \( A \) is indecomposable and that \( \tau_1 + \tau_2 = \tau_\infty \). If \( qa \in A(\tau_1) + A(\tau_2) \) for some prime \( q \) and some \( a \in A \), then \( a \in A(\tau_1) + A(\tau_2) \) since either \( qA(\tau_1) = A(\tau_1) \) or \( qA(\tau_2) = A(\tau_2) \). Consequently, \( A = A(\tau_1) + A(\tau_2) \), a completely decomposable group.

(c) First assume that \( A \) is indecomposable of rank 2, and that \( \tau_1 + \tau_2 = \tau_p \) for some prime \( p \). Then \( N_A = A/(A(\tau_1) + A(\tau_2)) \) is a cyclic group of order \( p^a \) for some \( 1 \leq a < \infty \) (see Theorem 1(a) and 1(b)). Let \( B \) be another rank 2 indecomposable \( R_A \)-group such that \( \text{typeset}(B) = \text{typeset}(A) \) where \( N_B \) is cyclic of order \( p^b \).
If $\alpha = \beta$ then there is a monomorphism $f: A \to B$ such that $f: A(\tau_1) \oplus A(\tau_2) \to B(\tau_1) \oplus B(\tau_2)$ is an isomorphism. Let $A = A(\tau_1) + A(\tau_2) + Zx$, where $x + A(\tau_1) + A(\tau_2)$ is a generator of $N_A$, and let $i$ be the least positive integer with $p^i f(x) \in B(\tau_1) \oplus B(\tau_2) = f(A(\tau_1) \oplus A(\tau_2))$. Since $f$ is monic, $i = \alpha$ and $f$ is an isomorphism. Thus $A \cong B$ iff $\alpha = \beta$.

Let $A = A_1 \oplus \cdots \oplus A_m = B_1 \oplus \cdots \oplus B_n$ be two decompositions of $A$ into indecomposable summands. Observe that $N_A$ is the direct sum of a finite number of copies of cyclic $p$-groups and a finite number of rank 1 groups of type $\tau_0$. It now follows that $m = n$ and $A_i \cong B_{\sigma(i)}$ for some permutation $\sigma$ of $\{1, 2, \cdots, n\}$ (use the preceding remarks and the fact that $N_A \cong N_{A_1} \oplus \cdots \oplus N_{A_m} \cong N_{B_1} \oplus \cdots \oplus N_{B_n}$).

**Proof of Theorem 2(a).** Assume that $A$ is indecomposable of rank $\geq 2$, denote $A(\tau_j)$ by $A_j$, $j = 1, 2, 3$, and write $A/(A_1 + A_2 + A_3) = C_1 \oplus \cdots \oplus C_k$, where $C_i$ is a cyclic group of order $m_i$ and $m_i$ divides $m_{i+1}$. Then $A = A_1 + A_2 + A_3 + Zx_1 + \cdots + Zx_k$, where $x_i + A_1 + A_2 + A_3$ is a generator of $C_i$, $m_i x_i = x_i + x_i + x_i$, and $x_i \in A_j$.

Define $S_j = \{x_{ji} \neq 0 | 1 \leq i \leq k\}$ and let $H_j$ be the pure subgroup of $A$ generated by $S_j$, where $1 \leq j \leq 3$. We prove that $X = \{x_1, \cdots, x_k\}$ may be chosen such that $H_j \cap A^* = 0$, where $\{j, k, l\} = \{1, 2, 3\}$ and $A^* = A$ is the pure subgroup of $A$ generated by $A_k + A_l$. Assume that, for example, $l_1 x_1 + \cdots + l_k x_k \in A^*_3$ for $l_i \in Z$ and that there is a least integer $t$ with $l_t x_{1t} \neq 0$. Then $\sum m_i l_i x_i \in A^*_3$ and $\sum r_i x_i \in A^*_3$ where, $r_i = (m_i l_i)/d$ and $d$ is the greatest common divisor of $\{m_1 l_1, \cdots, m_k l_k\}$. But $\sum A = \tau_\infty$ so there is an integer $n$ with $n(\sum r_i x_i) \in A_2 + A_3$ and $n A_1 = A_1$. Since $x_1 + A_1 + A_2 + A_3$ generates $C_1$, $m_i$ divides $n r_i$ for $1 \leq i \leq k$. But $r_i \neq 0$, consequently $m_i$ divides $n$, and $m_i A_i = A_i$. Define $x_i = m_i^{-1} (x_2 + x_3)$ a generator of $C_i$. Replacing $x_i$ by $x_i'$ and repeating the argument for all nonzero $l_i x_{1i}$ proves that $X$ may be chosen such that $H_1 \cap A^*_3 = 0$. Since replacing $x_i$ by $x_i'$ does not affect $S_2$ and $S_3$, similar arguments for $S_2$ and $S_3$ give the desired result. Note that, in fact, $S_i$ is a Z-independent subset of $A$ for $i = 1, 2, 3$.

Let $G_i$ be the pure subgroup of $A$ generated by $\{x_{1i}, x_{2i}, x_{3i}\}$. One easily proves that $G = G_1 \oplus G_2 \oplus G_3 \leq A$. Now $A_3 = A_3^* \oplus (G \cap A_3)$ and $A_3 \leq G \oplus A_3^* \leq A$. Furthermore, $A_2 = (G \cap A_2) \oplus (A_2' \cap A_2) \oplus A_2'$ and $A_2 + A_3 \leq G \oplus A_3' \oplus A_2'$. Finally, $A_1 = (G \cap A_1) \oplus (A_1' \cap A_1) \oplus (A_2' \cap A_1) \oplus A_1'$. But $x_1, \cdots, x_k \in G$, so $A = G_1 \oplus \cdots \oplus G_k \oplus A_3' \oplus A_2' \oplus A_1'$. Since $A$ is indecomposable, $A = G_k$, $k = 1$, and rank $A \leq 3$.

**Corollary 1.2.** Let $X_i$ be a rank 1 subgroup of $Q$ such that $1 \in X_i$ and type($X_i$) = $\tau_i$ for $i = 0, 1, 2, 3$ and let $V = Q_x \oplus Q_y \oplus Q_z$ be a $Q$-vector space of dimension 3. If $A$ is an indecomposable $R_\Lambda$-group and if $\sum A = \tau_\infty$, then $A$ is isomorphic to a group in one of the following classes of subgroups of $V$: [November]
(i) $X_0x, X_1x, X_2x, X_3x$;
(ii) $X_1x + X_2y + X_0((x+y)/n_3)$ where $2 \leq n_3 < \infty$ and $n_3 X_3 = X_3$;
(iii) $X_1x + X_3y + X_0((x+y)/n_2)$ where $2 \leq n_2 < \infty$ and $n_2 X_2 = X_2$;
(iv) $X_2x + X_3y + X_0((x+y)/n_1)$ where $2 \leq n_1 < \infty$ and $n_1 X_1 = X_1$;
(v) $X_1x + X_2y + X_3(x+y)$;
(vi) $X_1x + X_2y + X_3z + X_0((x+y+z)/n_1 n_2 n_3)$ where $2 \leq n_1 n_2 n_3 < \infty$ and $n_i X_i = X_i$.

**Lemma 1.3.** Let $A$ be an indecomposable $\Lambda$-group with typeset($A$) = $\Lambda$ and assume that if $p$ is a prime dividing the order of $A/(A(\tau_1) + A(\tau_2) + A(\tau_3))$, then $p A(\tau_i) = A(\tau_j)$ for some $j$. Then $A = A_{12}^* + A_{13}^* + A_{23}^*$ where $A_{ij}^*$ is the pure subgroup of $A$ generated by $A(\tau_i) + A(\tau_j)$.

**Proof.** Let $n$ be the smallest integer with $n A \subseteq A(\tau_1) + A(\tau_2) + A(\tau_3)$. The hypotheses guarantee the existence of relatively prime integers $n_1, n_2, n_3$ such that $n = n_1 n_2 n_3$ and $n_j A(\tau_j) = A(\tau_j)$.

Let $x = n^{-1}(a_1 + a_2 + a_3)$ be an arbitrary element of $A$ for some $a_j \in A(\tau_j)$. Choose integers $r_i$ and $s_i$ with $r_1 n_1 + s_1 n_2 = r_2 n_1 + s_2 n_3 = r_3 n_2 + s_3 n_3 = 1$. Then $x = u + v + w$, where

- $u = n_3^{-1}(n_1^{-1}r_3 a_1 + n_2^{-1}r_4 a_2) \in A_{12}^*$,
- $v = n_2^{-1}(n_1^{-1}s_3 a_1 + n_3^{-1}r_3 a_3) \in A_{13}^*$,
- $w = n_1^{-1}(n_2^{-1}s_2 a_2 + n_3^{-1}s_3 a_3) \in A_{23}^*$.

**Proposition 1.4.** Assume that $A$ is an indecomposable $\Lambda$-group with typeset($A$) = $\Lambda$; that $A(\tau_1) \oplus A(\tau_2) \oplus A(\tau_3) \subseteq A$; and that if $p$ is a prime dividing the order of $A/(A(\tau_1) \oplus A(\tau_2) \oplus A(\tau_3))$, then $p A(\tau_j) = A(\tau_j)$ for some $j$. Then $A \oplus A(\tau_1) \oplus A(\tau_2) \oplus A(\tau_3) 
\simeq A(\tau_1) \oplus A(\tau_2) \oplus A(\tau_3)$.

**Proof.** Let $A_j = A(\tau_j)$. It suffices to construct a split exact sequence

$0 \rightarrow A \xrightarrow{f_0} A/ A_1 \oplus A/ A_2 \oplus A/ A_3 \xrightarrow{f_1} A/ A_{12}^* \oplus A/ A_{13}^* \oplus A/ A_{23}^* \rightarrow 0$.

Note that $A/ A_{ij}^*$ is quasi-isomorphic, hence isomorphic, to $A_k$ where $\{i, j, k\} = \{1, 2, 3\}$ (recall that $A_k$ is homogeneous completely decomposable of type $\tau_k$).

Define $f_0$ by letting $f_0(a) = (a + A_1, -a + A_2, a + A_3)$ and define

$$f_1(a + A_1, b + A_2, c + A_3) = (a + b + A_{12}^*, c - a + A_{13}^*, -b - c + A_{23}^*).$$

Clearly $f_0$ and $f_1$ are well defined, $f_0$ is monic and image $f_0 \subseteq$ kernel $f_1$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
The image of \( f_0 \) is pure, for let \( p \) be a prime and let \( pw = f_0(a) \) where \( w = (x + A_1, y + A_2, z + A_3) \). Then \( px - a = a_1 \in A_1 \), \( py + a = a_2 \in A_2 \), and \( pz - a = a_3 \in A_3 \). Now \( p(x + 2y + z) = a_1 + 2a_2 + a_3 \in A_0 + A_4 \oplus A_3 \). If \( x + 2y + z \in A_1 + A_2 \oplus A_3 \), then \( p^{-1}a_j \in A(\tau_j) \), \( p^{-1}a \in A \) and \( w = f_0(p^{-1}a) \).

If

\[
x + 2y + z \notin A_1 \oplus A_2 \oplus A_3,
\]
then \( p^{-1}a_j \in A(\tau_j) \), for some \( j \) (recall hypotheses). Once again, \( p^{-1}a \in A \) and \( w = f_0(p^{-1}a) \).

The preceding remarks show that if kernel \( f_1/\text{image } f_0 \) is torsion, then kernel \( f_1 = \text{image } f_0 \). Let \( w = (a + A_1, b + A_2, c + A_3) \in \text{kernel } f_1 \). For some nonzero integer \( m \), \( m(a + b) = a_1 + a_2 \in A_1 + A_2 \), \( m(c - a) = b_1 + b_3 \in A_1 \oplus A_3 \) and \( m(-b - c) = c_2 + c_3 \in A_2 \oplus A_3 \). But \( 0 = m(a + b) + m(c - a) + m(-b - c) = (a_1 + b_1) + (a_2 + c_2) + (b_3 + c_3) \in A_1 + A_2 + A_3 \), so \( a_1 = -b_1 \), \( a_2 = -c_2 \) and \( b_3 = -c_3 \). Consequently, \( ma - a_1 = -mb + a_2 = mc - b_3 \) and \( f_0(ma - a_1) = (ma + A_1, mb + A_2, mc + A_3) = mw \). Thus kernel \( f_1/\text{image } f_0 \) is torsion, as desired.

To prove that \( f_1 \) is epic, let \( a + A_1^* \in A/A_1^* \) and write \( a = a_1^* + a_1^* + a_1^* \) where \( a_1^* \in A_1^* \). Then

\[
f_1(a_1^* + A_1, a_1^* + A_2, 0) = (a_1^* + a_1^* + A_2, -a_1^* + A_1^* - a_1^* + A_2^*),
\]

Similarly, \( A/A_1^* \) and \( A/A_2^* \) are contained in the image of \( f_1 \).

Finally, we define a homomorphism \( h: A/A_1^* \oplus A/A_2^* \oplus A/A_3^* \rightarrow A/A_1 \oplus A/A_2 \oplus A/A_3 \). Note that \( f_1: A_1^* \oplus A_2^* \oplus A_3^* \rightarrow A/A_1 \oplus A/A_2 \oplus A/A_3 \) is an epimorphism between two homogeneous completely decomposable groups of type \( \tau_3 \). Thus there is a homomorphism \( h_3: A/A_1^* \rightarrow A/A_1^* \) and \( h_2: A/A_2^* \rightarrow A/A_1^* \) with \( f_1h_3 = 1 \) (see Fuchs [4, p. 163] or [1]). Similarly there are maps \( h_3: A/A_1^* \rightarrow A/A_3^* \) and \( h_1: A/A_2^* \rightarrow A/A_2^* \) with \( f_1h_2 = 1 \) and \( f_1h_1 = 1 \). Define \( h = h_3 + h_2 + h_1 \), and observe that \( f_1h = 1 \). Thus (*) is a split exact sequence.

PROOF OF THEOREM 2(b).

A consequence of the fact that \( \sum \tau = \tau_\infty \) and Proposition 1.4.

PROOF OF COROLLARY 3. (a) If \( A \) is a \( \mathcal{C}_3 \)-group, then \( \text{typeset}(A) \subseteq \{ \tau_0 \cap \tau_0 \cap \tau_0, \tau_0 \cap \tau_0 \cap \tau_0 \cap \tau_0, \tau_0 \cap \tau_0 \cap \tau_0 \cap \tau_0 \cap \tau_0 \cap \tau_0 \} \). Suppose, for example, that \( \tau_0 \in \text{typeset}(A) \). Let \( R \) be a rank 1 subring of \( Q \) such that \( R \) has type \( \tau_0 \) (e.g., \( R = \mathbb{Z}_p \), the localization of \( \mathbb{Z} \) at \( p \)). Then \( A(\tau_0) \) is a free \( R \)-module, \( R \otimes \mathbb{Z} A(\tau_0) \cong A(\tau_0) \) and \( R \otimes \mathbb{Z} A = F \oplus D \), where \( F \) is a free \( R \)-module and \( D \) is divisible. It follows that \( R \otimes \mathbb{Z} A(\tau_0) \) is a summand of \( R \otimes \mathbb{Z} A \). Under the canonical imbedding, \( R \otimes \mathbb{Z} A(\tau_0) \subseteq A \subseteq R \otimes \mathbb{Z} A \) so that \( A(\tau_0) \) is a summand of \( A \). Thus \( A = A(\tau_0) \oplus A(\tau_0) \oplus A(\tau_0) \oplus G \) and \( \tau_0 \), \( \tau_0 \), \( \tau_0 \) \( \notin \text{typeset } G \).
(b) Let \( \tau_1 = \tau_p \cap \tau_q, \tau_2 = \tau_p \cap \tau_r, \tau_3 = \tau_q \cap \tau_r \). If \( B \) is an indecomposable rank 2 \( \mathcal{S}_3 \)-group, then \( B \) is in class (see Corollary 1.2)

- (ii) \( B/(B(\tau_1) \oplus B(\tau_2)) \cong Z/p^aZ, \quad 1 \leq a < \infty \);
- (iii) \( B/(B(\tau_1) \oplus B(\tau_3)) \cong Z/p^\beta Z, \quad 1 \leq \beta < \infty \);
- (iv) \( B/(B(\tau_2) \oplus B(\tau_3)) \cong Z/q^\gamma Z, \quad 1 \leq \gamma < \infty \); and
- (v) \( B/(B(\tau_3) \oplus B(\tau_1)) \cong Z(p^{\infty}) \) (the \( p \)-divisible subgroup of \( Q/Z \)).

Let \( A = G_1 \oplus \cdots \oplus G_k = H_1 \oplus \cdots \oplus H_l \) be a decomposition of \( A \) into indecomposable summands, where rank \( G_i \leq 2 \) and rank \( H_i \leq 2 \). Now \( A/(A(\tau_1) + A(\tau_2)) \) is a direct sum of a finite number of cyclic \( p \)-groups, a finite number of rank 1 groups of types \( \tau_p, \tau_q, \tau_r, \tau_0 \) and \( \tau_3 \) and a finite number of copies of \( Z(p^{\infty}) \). Moreover, \( A/(A(\tau_1) + A(\tau_3) + A(\tau_3)) \) is the direct sum of a finite number of cyclic \( p \)-groups, a finite number of cyclic \( q \)-groups, a finite number of cyclic \( r \)-groups, and a finite number of rank 1 groups of type \( \tau_p, \tau_q, \tau_r, \) and \( \tau_0 \). Thus \( k = l \) and \( G_i \cong H_{\sigma(i)} \) for some permutation \( \sigma \) of \( \{1, 2, \cdots, k\} \).

(c) A consequence of Theorem 2(b) and Corollary 3(b).

2. Examples. Let \( \Lambda = \{\tau_0, \tau_1, \tau_2, \tau_3\} \) where \( \tau_0 = \tau_1 \cap \tau_2 = \tau_1 \cap \tau_3 = \tau_2 \cap \tau_3 \); let \( \lambda = \{\tau_0, \tau_1, \tau_2\} \); let \( X_i \) be a rank 1 subgroup of \( Q \) with \( 1 \in X_i \) and with type \( (X_i) = \tau_i \); and let \( V = Qw \oplus Qx \oplus Qy \oplus Qz \).

Example 2.1. If \( \sum_i \neq \infty \), then there is a rank 2 indecomposable \( \mathcal{R} \)-group \( A \) with typeset \((A) = \lambda \).

Proof. Choose a prime \( p \) with \( pX_i \neq X_i \), where \( i = 1, 2 \), and let \( A = X_1w + X_2x + X_0((w+x)/p) \subset V \).

Example 2.2. If \( \sum_i \neq \tau_3 \) and \( \sum_i \neq \tau_1 \) for all primes \( p \), then there is an \( \mathcal{R} \)-group \( A \) with typeset \((A) = \lambda \) such that \( A = A_1 \oplus A_2 = A_3 \oplus B_1 \oplus B_2 \), where each \( A_i \) is an indecomposable group of rank 2 and \( B_1 \oplus B_2 \) is completely decomposable.

Proof. Choose distinct primes \( p \) and \( q \) with \( pX_i \neq X_i \) and \( qX_i \neq X_i \) for \( i = 1, 2 \). Define \( A_1 = X_1w + X_2x + X_0((w+x)/p) \), \( A_2 = X_1y + X_2x + X_0((y+x)/q) \) and let \( A = A_1 \oplus A_2 \). Let \( C_1 \) and \( C_2 \) be the pure subgroups of \( A \) generated by \( qw + px \) and \( qx + pz \), respectively. Define \( A_3 \) to be the pure subgroup of \( A \) generated by \( C_1 \oplus C_2 \) and note that \( A_3(\tau_1) = C_1 \oplus B_2 \) and \( A_3(\tau_2) = C_2 \oplus B_1 \), for some \( B_j \subset A(\tau_j) \). Clearly \( A_3 \cap (B_1 \oplus B_2) = 0 \) so that \( A(\tau_1) \oplus A(\tau_2) \subseteq A_3 \oplus B_1 \oplus B_2 \subseteq A \). But

\[
 a = p^{-1}(w + x) + q^{-1}(y + z) = ((qw + py) + (qx + pz))p^{-1}q^{-1} \in A_3
\]

and the order of \( a + A(\tau_1) + A(\tau_2) \) is \( pq \) in \( A/(A(\tau_1) \oplus A(\tau_2)) \) so that \( A = A_1 \oplus A_2 = A_3 \oplus B_1 \oplus B_2 \).

Example 2.3. If \( \sum_A \neq \tau_3 \), then there is an indecomposable rank 3 \( \mathcal{R} \)-group \( A \) such that typeset \((A) = \Lambda \) and \( A \oplus H \) is not a direct sum of
groups of rank \( \leq 2 \) for all finite rank completely decomposable \( R^\Lambda \)-groups \( H \).

**Proof.** Choose a prime \( p \) with \( pX_i \not\in X_i \) for \( i = 1, 2, 3 \) and let \( A = X_1w + X_2x + X_3y + X_0((w + x + y)/p) \). Note that \( A \) is indecomposable and \((w + x + y)/p \notin A_{12}^* + A_{13}^* + A_{23}^* \). If \( A \oplus H = G_1 \oplus \cdots \oplus G_k \), where \( H \) is a finite rank completely decomposable \( R^\Lambda \)-group and rank \( G_i \leq 2 \), then one can easily prove that \( A = A_{12}^* + A_{13}^* + A_{23}^* \), a contradiction.

**Example 2.4.** If \( \sum_{\Lambda} \not\in \tau \), then there is an indecomposable rank 3 \( R \)-group \( A \) such that typeset(\( A \)) = \( \Lambda \) and \( A \) is not quasi-isomorphic to a completely decomposable group (see Corollary 1.2).

**Proof.** Let \( A = X_1w + X_2x + X_3(y + x) \), a strongly indecomposable \( R^\Lambda \)-group of rank 2. Let \( A = A' + X_3y + X_0((w + x + y)/p) \) where \( p \) is a prime with \( pX_i \not\in X_i \) for \( i = 1, 2, 3 \). Then \( A \) is indecomposable of rank 3, \( A \) is quasi-isomorphic to \( A' \oplus X_3 \), and \( A' \) is not quasi-isomorphic to a completely decomposable group.

**References**


Department of Mathematical Sciences, New Mexico State University, Las Cruces, New Mexico 88001