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THE RADICAL OF $L^\infty(G)^*$ [1]

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ABSTRACT. THEOREM. *Let G be any locally compact nondiscrete group (or any infinite discrete amenable group). Then the radical of the (complex, noncommutative) Banach algebra $L^\infty(G)^*$ is not norm separable.*

Introduction. Let G be a locally compact group and denote by $\mathscr{R}$ the radical of the complex Banach algebra $L^\infty(G)^*$ with Arens multiplication induced from $L^1(G)$ (see next section). It has been proved by Civin and Yood [**2**] that if G is nondiscrete locally compact abelian (or G is the group of additive integers) then $\mathscr{R}$ is infinite dimensional. The conjecture of [**2**] that this holds true for all discrete abelian G has been proved by this author in [**5**, pp. 48, 58]; and, in fact, for any discrete infinite amenable group G, $\mathscr{R}$ is infinite dimensional [**5**]. It has been later proved by S. Gulick in [**8**, p. 136], among other results, that if G is any locally compact abelian group then $\mathscr{R}$ is not even norm separable. The following is posed as an unsolved question in [**8**, p. 136]: "If G is an infinite non-abelian locally compact group is the radical of $L^\infty(G)^*$ nonzero?" The theorem quoted in the abstract gives an incomplete answer to this question. The case it leaves out is for G an infinite discrete nonamenable group, for which the methods used in this paper fail. The method of proof resembles the one used by Gulick. We keep, however, a much closer account of the items involved.

Definitions and notations. Unless otherwise stated we adhere to the definitions of Hewitt-Ross [**9**]. $C(G)$ $[UCB_r(G)]$ will denote the space of bounded [right uniformly] continuous, complex, bounded functions with usual sup norm ($f\in UCB_r(G)$ iff $f\in C(G)$ and the map $x\to l_xf$, $l_xf(y)=f(xy)$, from G to $C(G)$ is continuous). Thus $UCB_r(G)\subset C(G)\subset L^\infty(G)$. The Arens multiplication is defined by: If $\mu,\nu\in L^\infty(G)$, $f\in L^\infty(G)$, $g\in L^1(G)$ then $f\odot g\in L^\infty(G)$, $\nu\odot f\in L^\infty(G)$, $\mu\odot\nu\in L^\infty(G)^*$ are defined by: $(f\odot g)(\phi)=(f, g*\phi)$, $(\nu\odot f)(\phi)=\nu(f\odot\phi)$ for all $\phi\in L^1(G)$ and

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$(\mu \odot \nu)f=\mu(\nu \odot f)$ for all $f\in L^\infty(G)$ (where (f, g) is the usual (L^∞, L^1) pairing). A topological space X is extremely [totally] disconnected if the closure $\bar{U}$ of any open set U is open [the connected component of each $x\in X$ is $\{x\}$ itself].

THEOREM. *Let G be a nondiscrete locally compact group. Then the radical of $L^\infty(G)^*$ is nonseparable.*

PROOF. Let $J=\{\psi\in L^\infty(G)^*;\ \psi[UCB_r(G)]=0\}$. We show at first that for all $\psi\in L^\infty(G)^*$, $\eta\in J$, $\psi\odot\eta=0$. Thus J will, in particular, be a left ideal such that $J^2=\{0\}$ and thus will be included in the radical $\mathscr{R}$ of $L^\infty(G)^*$ (see Rickart [**11**, p. 57, Theorem 2.3.5(ii)]).

Let $\eta\in J$, $\psi\in L^\infty(G)^*$, $f\in L^\infty(G)$, $g\in L^1(G)$. Then $f\odot g=(1/\Delta)g^\sim * f$ where $g^\sim(x)=g(x^{-1})$ (Wong [**12**, p. 354]). (Even though Wong assumes in [**12**] that L^1, L^∞, etc. are over the reals, the formulae on p. 352, Lemma 3.1.c of [**12**], upon which the proof of the last formula relies, hold true for complex valued f, ϕ, g.) Thus $f\odot g\in UCB_r(G)$ [**9**, p. 295, 20.16]. Hence $(\eta\odot f)(g)=\eta(f\odot g)=0$ for all $g\in L^1(G)$. Thus for all $f\in L^\infty(G)$, $\eta\odot f=0$. Hence $(\psi\odot\eta)(f)=\psi(\eta\odot f)=0$ which shows that $L^\infty(G)^*\odot J=\{0\}$. It can be shown (but is not needed here) that J is a two-sided ideal of $L^\infty(G)^*$. Now $J=UCB_r(G)^\perp\subset L^\infty(G)^*$ is the dual of the Banach space $L^\infty(G)/UCB_r(G)$ (with norm induced from $L^\infty(G)$). Therefore, whenever $L^\infty(G)/UCB_r(G)$ is nonseparable, J (and hence $\mathscr{R}$) will be such [**3**, II.3.16].

If G is not extremely disconnected then it is proved in (the proof of) Lemma 5.2, p. 134 of Gulick [**8**] that $L^\infty(G)/C(G)$ (and a fortiori $L^\infty(G)/UCB_r(G)$, see for example [**7**, p. 63]) is nonseparable. (Note that the commutativity of G is not used for this part in [**8**, p. 134].)

By the next proposition, any nondiscrete G is not extremely disconnected, which finishes this proof.

The next proposition is an improvement on a result of M. Rajagopalan [**10**] who proved it for locally compact abelian groups (see also Gulick [**8**, p. 126]).

PROPOSITION. *Let G be any locally compact extremely disconnected group. Then G is discrete.*[2]

PROOF. G is a fortiori totally disconnected and, by Hewitt-Ross [**9**, p. 62] contains some compact open subgroup $G_0\subset G$. G_0 is also extremely disconnected. It suffices to show that G_0 is discrete.

Let λ_0 be the normalized Haar measure of G_0. If G_0 is not discrete then

[2] Upon checking belatedly [**10**] we found that this proposition is due in its entirety to Rajagopalan.

$\lambda_0\{e\}=0$ where e is the identity. By regularity, there is a sequence of neighborhoods of e say V_n such that $\lambda_0(V_n)\to 0$. Let $K\subset \bigcap_1^\infty V_n$ be a normal closed subgroup such that G_0/K is metric [**9**, p. 71]. But G_0/K is also extremely disconnected. In fact if $\theta: G_0\to G_0/K$ is the canonical map then, for all $A\subset G_0$, $\theta(\bar{A})=\overline{\theta(A)}$, since $\theta(\bar{A})$ is compact and θ is continuous. If $U\subset G_0/K$ is open then so is $\theta^{-1}(U)$. Therefore

$$\theta(\overline{\theta^{-1}(U)}) = \overline{\theta(\theta^{-1}(U))} = \bar{U} \text{ is open}$$

since θ is an open map. Hence G_0/K is a metric extremely disconnected space (thus a metric F space [**4**, pp. 22 and 215]). By [**4**, p. 215] any metric F-space is discrete, which shows that G_0/K is finite, since G_0 is compact. Therefore $\lambda_0(K)>0$. But $\lambda_0(K)\leqq\lambda(V_n)\to 0$, which cannot be. Therefore G_0 (and hence G) is discrete.

REMARKS. We assume in what follows that G is discrete and amenable. It is shown in Day [**13**] that if μ, $\nu\in L^\infty(G)^*=UCB_r(G)^*$ one has $\mu\odot\nu(f)=\mu(F)$ where $F(x)=\nu(l_xf)$. It is assumed there that L^1, L^∞, $L^{\infty *}$ include only real-valued functions but the proof works unchanged for the complex case. Thus we are just in the case of the algebras $UCB_r(G)^*$ as dealt with in [**6**] or [**1**] (denoted there by $LUC(G)^*$).

For the real spaces L^∞, $L^{\infty *}$, it follows from Theorem 5.5 of Chou [**1**] (or from Theorem 5 of [**7**] combined with Day [**13**, p. 533, Theorem 2] and the lemma on p. 131 of [**6**]) that the radical of $L^\infty(G)^*$ is not norm separable.

For the complex spaces L^∞, $L^{\infty *}$ one argues as follows: Let $J_1=\{\psi\in L^\infty(G)^*;\ \psi(1)=0,\ l_a^*\psi=\psi \text{ for all } a\in G\}$. Then J_1 is a two-sided ideal with $J_1^2=\{0\}$. (See [**6**, p. 131]. The proof there works unchanged for the complex case.) Hence it is included in the radical of $L^\infty(G)^*$. If ϕ is a left invariant mean on the real real space $L^\infty(G)$ then extend ϕ uniquely to the complex $L^\infty(G)$ by $\phi(f)=\phi(\mathrm{Re}\, f)+i\phi(\mathrm{Im}\, f)$. Denote the set of such extended ϕ in the complex $L^\infty(G)^*$ by LIM. Pick some $\mu_0\in LIM$. Then $LIM-\mu_0=\{\mu-\mu_0;\ \mu\in LIM\}\subset J_1$. By Theorem 5.2 of Chou [**1**] the cardinality of LIM is at least 2^c if G is infinite, amenable (c=continuum). Thus J_1 is not norm separable, since a separable metric space contains at most c elements.

The reader interested in the radical of the algebra $l_1(S)^{**}$ for semigroups S is referred to [**14**].

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