Let $F_\pi$ denote the class of finite $p$-groups. It is well known that free groups and free polynilpotent groups are residually $F_\pi$ for all primes $p$. It is also known that such groups can be linearly ordered ([2, p. 49], [1, Theorem 4]). The purpose of this note is to prove the following general result on the orderability of residually $F_\pi$-groups.

**Theorem.** If $G$ is locally a residually $F_\pi$-group for infinitely many primes $p$, then $G$ can be linearly ordered.

To prove the Theorem we shall need the following result.

**Lemma.** Let $\alpha_{ij}$ ($i=1, \cdots, 2^n; j=1, \cdots, n$) be nonnegative integers such that $\sum_{j=1}^{2^n} \alpha_{ij}>0$ for all $i$. Let $p$ be a prime such that $p \geq (2\alpha)^{2^n}$ where $\alpha=\max\{\alpha_{ij}\}$. Then in any solution set to the system of $2^n$ equations

$$
\sum_{j=1}^{2^n} \varepsilon_{ij} \alpha_{ij} x_j \equiv 0 \pmod{p}
$$

for some $j$. Assume the result holds for $n-1$ and let $n=m$. Let $d=2^{m-1}$ and for any integer $r$ let $r'=d+r$. By renumbering the $2^m$ equations, if necessary, assume that $\varepsilon_{i1}=1$, $\varepsilon_{r1}=-1$ and $\varepsilon_{ij}=\varepsilon_{rj}$ for all $i \in \{1, \cdots, d\}$ and $j>1$. We produce a system of $d$ equations in $m-1$ unknowns satisfying the hypotheses of the Lemma as follows. For any $r \in \{1, \cdots, d\}$ we have

$$
\alpha_{r1} x_1 + \sum_{j=2}^{m} \varepsilon_{rj} \alpha_{rj} x_j \equiv 0 \pmod{p},
$$

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\[ -\alpha_r x_1 + \sum_{j=2}^{m} \varepsilon_{ij} \alpha_{rj} x_j \equiv 0 \pmod{p}. \]

As the \( r \)th equation of the new system pick: I if \( \alpha_{r1} = 0 \); II if \( \alpha_{r1} \neq 0 \) but \( \sum_{j=2}^{m} \varepsilon_{ij}(\alpha_{r1}\alpha_{rj} + \alpha_{r1}\alpha_{rj})x_j \equiv 0 \pmod{p} \) if \( \alpha_{r1} \neq 0 \) and \( \alpha_{r1} \neq 0 \). Notice that for any \( r \) if \( \sum_{j=2}^{m} \alpha_{rj} = 0 \) then \( x_1 \equiv 0 \pmod{p} \) and the Lemma is proved. Thus we may assume that at least one coefficient of the \( r \)th equation of the new system is positive. Let \( \beta \) be the maximum of the coefficients of the new system of equations. Then \( \beta \leq (2\alpha)^2 \) so that \( p \geq (2\alpha)^{2^n} = (4\alpha^2)^{2^{m-1}} \geq (2\beta)^{2^{m-1}} \). Thus the hypotheses of the Lemma are satisfied for this system and, by induction, \( x_j \equiv 0 \pmod{p} \) for some \( j \).

**Proof of the Theorem.** We use a characterization of ordered groups given by Fuchs [2, p. 34]. If \( x \) is an element of a group \( G \) and \( \alpha \) a non-negative integer, we write \([x]^\alpha \) to denote a product of \( \alpha \) conjugates of \( x \) in \( G \).

Suppose, if possible, that the Theorem is false. Then there exist elements \( x_i \in G \) for \( i = 1, \ldots, n \) in \( G \) such that

\[ [x_i^{\varepsilon_{ij}}]^{a_1} \cdots [x_n^{\varepsilon_{ijn}}]^{a_n} = e \]

for \( 2^n \) values of \( i \) associated with signs \( \varepsilon_{ij} \equiv \pm 1 \). Moreover \( \sum_{i=1}^{n} \alpha_{ij} > 0 \) for all \( i \). Assume that \( n \) is the least such integer. Let \( \alpha = \max \{ \alpha_{ij} \} \). We can assume that \( G \) is generated by the finite set of elements involved in the equations given by (1), so that \( G \) is residually \( F_p \) for infinitely many \( p \). Choose a prime \( p > (2\alpha)^{2^n} \) such that \( G \) is residually \( F_p \) for this prime. Let \( X = \langle x_1^{a_1}, \ldots, x_n^{a_n} \rangle \), the normal subgroup of \( G \) generated by \( x_1, \ldots, x_n \). There exists \( K \triangleleft G \) maximal subject to \( G/K \in F_p, X \subseteq K \).

Let \( \theta \) be the homomorphism \( G \rightarrow G/K = G^* \), and let \( \theta(x_i) = a_i, \theta(X) = A \). Then \( G^* \in F_p, A \neq \{e\} \), and, by our choice of \( K, A \) is a minimal normal subgroup of \( G^* \). Thus \( A \) is cyclic of order \( p \) in the centre of \( G^* \). By reordering the suffixes, if necessary, assume that \( A = \langle a_1 \rangle, a_j = a_j^\beta_j \) where \( \beta_1 = 1, 0 < \beta_j < p \) for \( j = 2, \ldots, m \) and \( \beta_j = 0 \) for \( j > m \). The \( 2^n \) equations given by (1) now reduce to the following.

\[ a_1^{(\sum_{j=1}^{m} \varepsilon_{ij} \alpha_{ij} \beta_j)} = e, \quad j = 1, \ldots, m, i = 1, \ldots, 2^n. \]

If \( m < n \), then from the \( 2^n \) equations above choose \( 2^m \) equations associated with the \( 2^m \) choices of signs \( \varepsilon_{ij} = \pm 1 \) \((j = 1, \ldots, m)\) in such a way that for each \( i \in \{1, \ldots, 2^m\} \) at least one \( \alpha_{ij} \neq 0 \) in this set of \( 2^m \) equations. This is possible by our choice of \( n \). The above equations now reduce to

\[ \sum_{j=1}^{m} \varepsilon_{ij} \alpha_{ij} \beta_j \equiv 0 \pmod{p}, \quad i = 1, \ldots, 2^m. \]
The above equations satisfy the hypotheses of the Lemma and hence
\[ \beta_j \equiv 0 \pmod{p} \] for some \( j \in \{1, \ldots, m\} \). This is a contradiction and the
Theorem is proved.

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