PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 41, Number 1, November 1973

CONTINUOUS FUNCTIONS INDUCED BY SHAPE MORPHISMS[1]

JAMES KEESLING

ABSTRACT. Let C denote the category of compact Hausdorff spaces and continuous maps and $H:C\to HC$ the homotopy functor to the homotopy category. Let $S:C\to SC$ denote the functor of shape in the sense of Holsztyński for the projection functor H. Every continuous mapping f between spaces gives rise to a shape morphism $S(f)$ in SC, but not every shape morphism is in the image of S. In this paper it is shown that if X is a continuum with $x\in X$ and A is a compact connected abelian topological group, then if F is a shape morphism from X to A, then there is a continuous map $f:X\to A$ such that $f(x)=0$ and $S(f)=F$. It is also shown that if $f,g:X\to A$ are continuous with $f(x)=g(x)=0$ and $S(f)=S(g)$, then f and g are homotopic. These results are then used to show that there are shape classes of continua containing no locally connected continua and no arcwise connected continua. Some other applications to shape theory are given also.

Introduction. Let C denote the category of compact Hausdorff spaces and continuous maps and $H:C\to HC$ the homotopy functor to the homotopy category. Let $S:C\to SC$ denote the functor of shape in the sense of Holsztyński for the projection functor H [5]. Let X and Y be compact Hausdorff spaces. In [6] it is shown that if X and Y are associated with ANR-systems $\underline{X}$ and $\underline{Y}$, respectively, then there is one to one correspondence between $\mathrm{Mor}_{SC}(X, Y)$ and the homotopy classes of maps of ANR-systems used in the approach of Mardesič and Segal [7]. Thus, our results in this paper will apply to either approach to shape.

In the first part of the paper we show that if X is a continuum with $x\in X$ and A is a compact connected abelian topological group, then if $F\in \mathrm{Mor}_{SC}(X, A)$, then there is a continuous $f:X\to A$ with $S(f)=F$ and with $f(x)=0$. It is also shown that if X and A are as above and $f, g:X\to A$ are continuous with $f(x)=g(x)=0$ and with $S(f)=S(g)$, then f and g are homotopic. These results are clearly related to the results in [6].

Received by the editors November 14, 1972.

AMS (MOS) subject classifications (1970). Primary 55D99; Secondary 22B99.

Key words and phrases. Shape functor, homotopy functor, shape domination, compact connected abelian topological group, locally connected continuum, arcwise connected continuum.

[1] This research was supported by NSF Grant GP-24616A#1.

In the second part of the paper we give applications of these results. It is shown that if A is a compact connected abelian topological group, then if A is not locally connected, and if X shape dominates A, then X is not locally connected. It is also shown that if $F \in \mathrm{Mor}_{SC}(G, A)$ where G is a compact connected topological group, then there is a unique continuous homomorphism $f: G \to A$ with $S(f) = F$.

Notation. We assume the notation and results of [**6**]. As in that paper it is assumed that the reader is familiar with the approach of Holsztyński to shape theory [**5**] and the approach of Mardešič and Segal [**7**]. A knowledge of topological groups is assumed. Pontrjagin [**8**] is a good reference. We will use the fact shown in [**1**] that if G is a T_1 topological group and L is a subgroup of G which is a compact Lie group, then the quotient map $p: G \to G/L$ is a fibration. Now p can be thought of as the composition of two maps, $p_0: G \to G/L_0$, where L_0 is the component of the identity e in L, and $p_1: G/L_0 \to G/L$. The map p_0 is monotone and p_1 is a covering map which is finite-to-one.

The cardinals are thought of as a subset of the ordinals as initial ordinals in the usual way. Infinite cardinals are denoted by ω_α where α is an ordinal and ω_α is the αth infinite cardinal. Thus, ω_0 is the first infinite cardinal, ω_1 is the first uncountable cardinal, and so on. If X is a topological space, then the weight of X, $w(X)$, is the minimum cardinality of a basis for X.

1. **Induced mappings.** Our main theorems are proved in this section. Basically we show that there is a one-to-one correspondence between $\mathrm{Mor}_{SC}(X, A)$ and $\{H(f): f \in \mathrm{Mor}_C(X, A)$ and $f(x) = 0\} \subset \mathrm{Mor}_{HC}(X, A)$ where X is a continuum with $x \in X$ and A is a compact connected abelian topological group. More precisely, Theorems 1.1 and 1.2 taken together imply that if $F: HC \to SC$ is the functor such that $S = F \circ H$, then for $f, g \in \mathrm{Mor}_C(X, A)$, then $S(f) = S(g)$ if and only if $H(L_{-f(x)} \circ f) = H(L_{-g(x)} \circ g)$ where $L_a: A \to A$ is left translation by $a \in A$.

1.1. THEOREM. *Let F be a morphism in SC from X to A where X is a continuum with $x \in X$ and A is a compact connected abelian topological group. Then there is a continuous function $f: X \to A$ with $f(x) = 0$ and with $S(f) = F$.*

1.2. THEOREM. *Suppose that X and A are as in Theorem 1.1. Then if $f, g: X \to A$ are continuous with $f(x) = g(x) = 0$, then $S(f) = S(g)$ implies $H(f) = H(g)$.*

The key to the proofs of these theorems is Theorem 68 of [**8**, p. 326] which we state before beginning the proofs of the above theorems.

1.3. DEFINITION. Let $\mathscr{G}=\{G_\alpha;\ \pi_{\alpha\beta};\ \alpha\leqq\beta<\gamma\}$ be an inverse system of compact topological groups with $\pi_{\alpha\beta}(G_\beta)=G_\alpha$ continuous homomorphisms. Assume that the indexing set is the ordinals $[0,\gamma)$ with the usual ordering. Then $\mathscr{G}$ is a *Lie series* provided that (1) G_0 is a Lie group; (2) $\ker \pi_{\alpha\,\alpha+1}$ is a Lie group for all $\alpha<\gamma$; and (3) for every limit ordinal $\beta<\gamma$, $\bigcap_{\alpha<\beta}\ker \pi_{\alpha\beta}=\{e\}$ in G_β.

1.4. THEOREM. [**8**, Theorem 68, p. 326]. *Suppose that G is a compact topological group with $w(G)=\omega_\gamma$. Then there is a Lie series $\{G_\alpha;\ \pi_{\alpha\beta};\ \alpha\leqq\beta<\omega_\gamma\}$ having G as inverse limit.*

Observe that if $w(G)>\omega_0$, then $w(G_\alpha)<w(G)$ for all $\alpha<\omega_\gamma$ in the Lie series. It will be convenient to prove Theorem 1.2 first and then prove Theorem 1.1.

PROOF OF THEOREM 1.2. Clearly Theorem 1.2 is true if A is a torus, since A is then an ANR and any two functions having the same shape are homotopic. Now suppose that $w(A)=\omega_\gamma$ and let $\{A_\alpha;\ \pi_{\alpha\beta};\ a\leqq\beta<\omega_\gamma\}$ be a Lie series having A as inverse limit. Of course, all A_α's are abelian in this case since A is abelian. Note that for all $\alpha<\omega_\gamma$, $S(\pi_\alpha\circ f)=S(\pi_\alpha\circ g)$ and $\pi_\alpha\circ f(x)=\pi_\alpha\circ g(x)=0$ in A_α where $\pi_\alpha: A\to A_\alpha$ are the projection maps making A the inverse limit of the Lie series. Now $\pi_0\circ f$ and $\pi_0\circ g$ are homotopic by the above remark. Let $J_0: X\times I\to A_0$ be a homotopy with $J_0|X\times\{0\}\equiv\pi_0\circ f$ and $J_0|X\times\{1\}\equiv\pi_0\circ g$ such that $J_0(\{x\}\times I)=0\in A_0$. Suppose now that for $0<\beta<\omega_\gamma$ we have homotopies $J_\alpha: X\times I\to A_\alpha$ defined for all $\alpha<\beta$ such that (1) $J_\alpha|X\times\{0\}\equiv\pi_\alpha\circ f$ and $J_\alpha|X\times\{1\}\equiv\pi_\alpha\circ g$; (2) such that $J_\alpha(\{x\}\times I)=0\in A_\alpha$; and (3) such that for all $\alpha\leqq\delta<\beta$, $J_\alpha=\pi_{\alpha\delta}\circ J_\delta$. Then if β is a limit ordinal, then let J_β be the limit of the maps $J_\alpha: X\times I\to A_\alpha$ in the inverse system $\{A_\alpha;\pi_{\alpha\delta};\alpha\leqq\delta<\beta\}$. Then $\{J_\alpha:\alpha\leqq\beta\}$ has properties (1), (2), and (3) above. If $\beta=\alpha+1$, then let $\pi_{\alpha\,\alpha+1}=p\circ q$ where $q: A_{\alpha+1}\to A'$ is a monotone homomorphism and $p: A'\to A_\alpha$ is a finite-to-one covering map so that $p\circ q$ is the monotone-light factorization of $\pi_{\alpha\,\alpha+1}$. Note that the kernel of q is a torus in $A_{\alpha+1}$.

CLAIM. *Let B be a subgroup of A with A a compact connected abelian topological group and B a torus. Then $A\simeq B\times A/B$.*

PROOF OF CLAIM. Let $e: B\to A$ denote the imbedding of B as a subgroup. Then $e^*:\operatorname{char} A\to\operatorname{char} B$ is onto. Now $\operatorname{char} B=Z^n$ is free. Thus $\operatorname{char} A\simeq Z^n\times\ker e^*$. Thus $A\simeq B\times C$ where $C=\operatorname{char}\ker e^*$. Clearly $C\simeq A/B$.

The claim allows us to view $A_{\alpha+1}$ as $A'\times\ker q$. Since p is a covering map, there is a unique lifting of J_α to $J': X\times I\to A'$ such that $J'(\{x\}\times I)=0\in A'$ and $J'|X\times\{0\}=q\circ\pi_{\alpha+1}\circ f$ and $J'|X\times\{1\}=q\circ\pi_{\alpha+1}\circ g$. Let $r: A_{\alpha+1}\to\ker q$ be the projection map. Since $\ker q$ is a torus and $S(r\circ\pi_{\alpha+1}\circ f)=S(r\circ\pi_{\alpha+1}\circ g)$ with $r\circ\pi_{\alpha+1}\circ f(x)=r\circ\pi_{\alpha+1}\circ g(x)=0\in\ker q$, there is a

homotopy $J'':X\times I\to\ker q$ such that $J''|X\times\{0\}=r\circ\pi_{\alpha+1}\circ f$ and $J''|X\times\{1\}=r\circ\pi_{\alpha+1}\circ g$ with $J''(\{x\}\times I)=0\in\ker q$. Then let $J_{\alpha+1}:X\times I\to A_{\alpha+1}$ be defined by $J_{\alpha+1}(x,t)=(J'(x,t),J''(x,t))\in A'\times\ker q=A_{\alpha+1}$. Then in this case also $\{J_\alpha:\alpha\leqq\beta\}$ has properties (1), (2), and (3) above. Proceeding by transfinite induction we get a collection of maps $\{J_\alpha:\alpha<\omega_\gamma\}$ satisfying properties (1), (2), and (3). Then letting $J:X\times I\to A$ be the limit of $\{J_\alpha:\alpha<\omega_\gamma\}$ J will be a homotopy from f to g and Theorem 1.2 is proved.

PROOF OF THEOREM 1.1. The proof of Theorem 1.1 proceeds by transfinite induction on $w(A)$. If A is a torus, then clearly there is such a continuous $f:X\to A$ since A is an ANR. We can require $f(x)=0$ by using left multiplication in A. Suppose now that $w(A)=\omega_0$. Then let A be the inverse limit of a sequence of Lie groups $\{A_i;\pi_{ij};i\leqq j<\omega_0\}$. Of course, the A_i's are tori in this case. By [1], each map $\pi_{i\,i+1}$ is a fibration. By the remarks above, $S(\pi_i)\circ F=S(f_i)$ for some $f_i:X\to A_i$ with $f_i(x)=0$ for each i where $\pi_i:A\to A_i$ are the projection maps making A the limit of $\{A_i\}$. Let $g_1=f_1$. Now $S(g_1)=S(\pi_1)\circ F=S(\pi_{12})\circ S(\pi_2)\circ F=S(\pi_{12}\circ f_2)$. By Theorem 1.2, g_1 and $\pi_{12}\circ f_2$ are homotopic. Since π_{12} is a fibration the map g_1 can be lifted to a map g_2 such that g_2 is homotopic to f_2 with $g_2(x)=0$. By the same argument g_2 can be lifted to a map $g_3:X\to A_3$ such that g_3 is homotopic to f_3 with $g_3(x)=0$. Proceeding by induction we get a sequence of maps $g_i:X\to A_i$ with $g_i(x)=0$ and with $g_i=\pi_{ij}\circ g_j$ for all $i\leqq j<\omega_0$ with f_i and g_i homotopic. Thus $S(g_i)=S(\pi_i)\circ F$. Let $f:X\to A$ be the limit of the maps $\{g_i\}$. Then $S(f)=F$ by the continuity of the shape functor [5]. Clearly $f(x)=0$.

Now suppose that $w(A)=\omega_\gamma>\omega_0$ and assume that for all $\omega_\alpha<\omega_\gamma$ if $w(A)=\omega_\alpha$, then the theorem is true for A. Then let $\{A_\alpha;\pi_{\alpha\beta};\alpha\leqq\beta<\omega_\gamma\}$ be a Lie series having A as inverse limit. Note that $w(A_\alpha)<\omega_\gamma$ for all $\alpha<\omega_\gamma$. Thus there are maps $f_\alpha:X\to A_\alpha$ with $f_\alpha(x)=0\in A_\alpha$ and $S(f_\alpha)=S(\pi_\alpha)\circ F$ for all $\alpha<\omega_\gamma$. Let $g_0=f_0$ and suppose that g_α has been defined for all $\alpha<\beta<\omega_\gamma$ such that $g_\alpha:X\to A_\alpha$ is homotopic to f_α with $g_\alpha(x)=0\in A_\alpha$ and with $g_\alpha=\pi_{\alpha\delta}\circ g_\delta$ for all $\alpha\leqq\delta<\beta$. Then if β is a limit ordinal let $g_\beta:X\to A_\beta$ be the limit of the maps $\{g_\alpha:\alpha<\beta\}$. Then $g_\beta(x)=0$ and $S(g_\beta)=S(\pi_\beta)\circ F=S(f_\beta)$ by the continuity of the shape functor. Thus by Theorem 1.2, f_β and g_β are homotopic. Now if $\beta=\alpha+1$, then by [1], $\pi_{\alpha\,\alpha+1}$ is a fibration with g_α homotopic to $\pi_{\alpha\beta}\circ f_\beta$. Thus g_α can be lifted to a map g_β homotopic to f_β with $g_\beta(x)=0$. Thus proceeding by transfinite induction we obtain a collection of maps $\{g_\alpha:\alpha<\omega_\gamma\}$ with $g_\alpha:X\to A_\alpha;g_\alpha(x)=0$; and $S(g_\alpha)=S(\pi_\alpha)\circ F$. Letting f be the limit of these maps, we get that $f(x)=0$ and $S(f)=F$. Thus the theorem is true for $w(A)=\omega_\gamma$. Theorem 1.1 now follows by transfinite induction on $w(A)$.

It is not known if Theorems 1.1 and 1.2 would be true for A an

arbitrary compact connected topological group. However, Theorem 1.1 can be shown for this case if $w(A)=\omega_0$.

1.5. THEOREM. *Let X be a continuum and $x \in X$. Let A be a compact connected topological group with identity element e and with $w(A)=\omega_0$. Then, if $F \in \mathrm{Mor}_{SC}(X, A)$, there is a continuous map $f: X \to A$ with $f(x)=e$ and with $S(f)=F$.*

PROOF. Let $\{A_i; \pi_{ij}; i \leqq j < \omega_0\}$ be a sequence of Lie groups having A as inverse limit with the projection maps being $\pi_i: A \to A_i$. For each i, there is a continuous map $f_i: X \to A_i$ with $f(x)=e \in A_i$ such that $S(f_i)=S(\pi_i) \circ F$ since each A_i is an ANR. Using the fact that $\pi_{i\ i+1}$ is a fibration for each i and that f_i is homotopic to $\pi_{i\ i+1} \circ f_{i+1}$ for each i, construct $g_i: X \to A_i$ with $g_i(x)=e \in A_i$ with g_i homotopic to f_i and with $g_i = \pi_{ij} \circ g_j$ for all $i \leqq j < \omega_0$. Then let f be the limit of the g_i's. Then $S(f)=F$ by the continuity of the shape functor.

2. **Applications.** In this section we apply Theorem 1.1 to obtain several theorems and examples in shape theory.

2.1. THEOREM. *Let A be a compact connected abelian topological group and X a continuum. Suppose that $S(X) \geqq S(A)$. Then there is a continuous map $f: X \to A$ such that $f(X)=A$.*

PROOF. Suppose that $S(X) \geqq S(A)$. Then there are shape morphisms $F: X \to A$ and $G: A \to X$ such that $F \circ G = 1_A$ in SC. By Theorem 1.1 there is a continuous map $f: X \to A$ such that $S(f)=F$. We will now show that $f(X)=A$. Suppose that $f(X) \neq A$. Let $e: f(X) \to A$ be the inclusion map. Now A is a limit manifold [**3**, 2.12, p. 345]. Thus $e^*: H^*(A) \to H^*(f(X))$ is not a monomorphism (the proof of Proposition 2.11 in [**3**, p. 344]). But then $f^*: H^*(A) \to H^*(X)$ cannot be a monomorphism either. But then $G^* \circ F^* = G^* \circ f^*: H^*(A) \to H^*(A)$ cannot be $1_{H^*(A)}$, a contradiction. Thus $f(X)=A$ as asserted.

2.2. COROLLARY. *Let A be a compact connected abelian topological group. Let X be a continuum. If $S(X) \geqq S(A)$ and X is locally connected, then A is locally connected. If $S(X) \geqq S(A)$ and X is arcwise connected, then A is arcwise connected.*

Actually Corollary 2.2 is true for any property $\mathscr{P}$ such that if X has property $\mathscr{P}$ and $f(X)=Y$ is a quotient map, then Y has property $\mathscr{P}$.

2.3. EXAMPLE. Let Σ_a be a solenoid which is not a circle. Then for any continuum X with $S(X) \geqq S(\Sigma_a)$, X cannot be arcwise connected or locally connected. Actually there must be at least 2^{ω_0} arccomponents of

X because of the map $f(X)=\Sigma_a$. Borsuk [**0**] proved that no (metric) Peano continuum can shape dominate a solenoid.

2.4. EXAMPLE. Let Z be the integers and let $A=\operatorname{char} Z^{\omega_0}$. It is known that A is a locally connected connected abelian topological group which is not arcwise connected. For a proof of this last fact see [**2**, 4.82, p. 194]. The space A is an example of a locally connected continuum such that no arcwise connected continuum shape dominates A. In particular, no arcwise connected continuum has the same shape as A. This was mentioned without proof in [**6**]. In [**6**] it was shown that A is movable since it is locally connected.

2.5. THEOREM. *Let G be a compact connected topological group and A a compact connected abelian topological group. Then if $F \in \operatorname{Mor}_{SC}(G, A)$, then there is a unique continuous homomorphism $f: G \to A$ such that $S(f)=F$.*

PROOF. Let $F \in \operatorname{Mor}_{SC}(G, A)$. Then by Theorem 1.1, there is a continuous $g: G \to A$ such that $g(e)=0$ with $S(g)=F$. By [**9**], g is homotopic to a continuous homomorphism $f: G \to A$. Thus $S(f)=F$ also. If $f': G \to A$ is a continuous homomorphism with $S(f')=F$, then f' and f are homotopic by Theorem 1.2. But by [**9**], $f'=f$ and Theorem 2.5 is proved.

Theorem 2.5 generalizes Theorem 1.2 in [**6**].

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32601.