PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 41, Number 1, November 1973

DIFFERENTIABLE PROJECTIONS AND DIFFERENTIABLE SEMIGROUPS

J. P. HOLMES

ABSTRACT. Suppose X is a Banach space, G is a connected open subset of X, and p is a continuously Fréchet differentiable function from G into G satisfying $p(p(x))=p(x)$ for each x in G. We prove that $p(G)$ is a differentiable submanifold of X and use this result to show that the maximal subgroup containing an idempotent in a differentiable semigroup is a Lie group.

Suppose X is a Banach space, G is a connected open subset of X, and p is a continuously Fréchet differentiable function from G into G satisfying $p(p(x))=p(x)$ for each x in G. We prove that $p(G)$ is a differentiable submanifold of X and use this result to show that the maximal subgroup containing an idempotent in a differentiable semigroup is a Lie group.

In [4] Nadler proved that the image of a differentiable projection defined on an open subset of a Euclidean space is a differentiable manifold.

THEOREM 1. *Suppose each of X, p, and G is as above. There is a closed linear subspace Y of X and a set $H=\{h_x\}$ of functions, one for each x in $p(G)$ so that, for each x in $p(G)$, h_x is a homeomorphism from a neighborhood of x in $p(G)$ onto a neighborhood of* 0 *in Y, $h_x(x)=0$, and if each of x and y is in $p(G)$ with y in* $\operatorname{dom}(h_x)$ *then there is a neighborhood U_{xy} of* 0 *in Y so that $h_x \circ h_y^{-1}$ is continuously differentiable on U_{xy}.*

First, we establish some notation. Since $p^2=p$ we have, by the chain rule, that $p'(p(x)) \circ p'(x)=p'(x)$ for each x in G. In particular, if x is in $p(G)$ then $p'(x)$ is a continuous linear idempotent mapping from X into X. Denote by Y_x the image of $p'(x)$ if x is in $p(G)$. Let p_x denote the function defined from $G-x$ to $G-x$ by $p_x(y)=p(y+x)-x$ for each x in $p(G)$. Finally, if d is a positive number denote by $R(d)$ the subset of X to which x belongs if and only if $\|x\|<d$.

LEMMA 1. *If x is in $p(G)$ then there is a $d_x>0$ so that $p'(x)$ is one-to-one on $R(d_x)\cap p_x(G-x)$.*

Received by the editors January 18, 1973 and, in revised form, February 26, 1973.
AMS (MOS) subject classifications (1970). Primary 52B99; Secondary 22E65.
Key words and phrases. Differentiable manifold, differentiable semigroup.

PROOF. Choose d_x by continuity of p'_x so that if y is in $R(d_x)$ then y is in $G-x$ and $\|p'_x(y)-p'_x(0)\|<\frac{1}{2}$.

Suppose that each of y and z is in $R(d_x)\cap p_x(G-x)$.

$$\begin{aligned}\|y-z\| &= \|p_x(y)-p_x(z)\| \\ &\leqq \|p_x(y)-p_x(z)-p'_x(0)(y-z)\| + \|p'_x(0)(y-z)\| \\ &= \left\|\int_0^1 dt[p'_x(z+t(y-z))-p'_x(0)](y-z)\right\| + \|p'_x(0)(y-z)\| \\ &\leqq \tfrac{1}{2}\|y-z\| + \|p'_x(0)(y-z)\|.\end{aligned}$$

Thus $\frac{1}{2}\|y-z\|\leqq\|p'_x(0)(y-z)\|$ and we are done.

LEMMA 2. *Suppose x is in $p(G)$. There are neighborhoods U_x and V_x of* 0 *in Y_x so that $f_x=p'(x)\circ p_x$ satisfies*:

(a) *$(f_x|U_x)$ is a homeomorphism onto V_x.*

(b) *Each of U_x and $p_x(U_x)$ is contained in $R(d_x)$.*

(c) *$(f_x|U_x)^{-1}$ is continuously differentiable on V_x.*

PROOF. This is a simple application of the inverse function theorem [2, p. 268] and the observation that $(f_x|Y_x\cap(G-x))'(0)$ is the identity function on Y_x.

LEMMA 3. *If x is in $p(G)$ then $(p_x|U_x)$ is a homeomorphism onto a neighborhood of* 0 *in $p_x(G-x)$.*

PROOF. By Lemma 2, p_x is one-to-one on U_x. Choose e positive and less than d_x so that $p'(x)(R(e))$ is contained in V_x. If y is in $R(e)\cap p_x(G-x)$ then, by Lemma 2, there is a z in $p_x(U_x)$ so that $p'(x)(z)=p'(x)(y)$. Since $p_x(U_x)$ is contained in $R(d_x)$ and $e<d_x$ we have $y=z$ is in $p_x(U_x)$. Hence $p_x(U_x)$ is a neighborhood of 0 in $p_x(G-x)$.

Let $g_x=(p_x|U_x)^{-1}$ and suppose each of y and z is in $B_x=\mathrm{dom}(g_x)$.

$$\begin{aligned}&\|g_x(y)-g_x(z)\|-\|y-z\| \\ &\quad\leqq \|g_x(y)-g_x(z)-(y-z)\| \\ &\quad= \|p_x(g_x(y))-p_x(g_x(z))-p'(x)(g_x(y)-g_x(z))\| \\ &\quad= \left\|\int_0^1 dt[p'_x(g_x(z)+t(g_x(y)-g_x(z)))-p'_x(0)](g_x(y)-g_x(z))\right\| \\ &\quad\leqq \tfrac{1}{2}\|g_x(y)-g_x(z)\|\end{aligned}$$

since each of $g_x(y)$ and $g_x(z)$ is in $R(d_x)$. Thus $\|g_x(y)-g_x(z)\|\leqq 2\|y-z\|$ and we are done.

For each x in $p(G)$ let $C_x=B_x+x$ and define m_x on C_x by $m_x(y)=g_x(y-x)$. C_x is a neighborhood of x in $p(G)$ and m_x is a homeomorphism from C_x onto U_x.

LEMMA 4. *Suppose each of x and y is in $p(G)$ and y is in C_x. Then there is a $d(x, y)>0$ such that $(m_x \circ m_y^{-1}|R(d(x, y))\cap U_y)$ is continuously differentiable.*

PROOF. If y is in C_x then $y-x$ is in B_x. $p_x(y-x)=y-x$ so by continuity of p_x and $+$ there is a $d(x, y)>0$ so that each of $R(d(x, y))$ and $y-x+R(d(x, y))$ is contained in $G-x$ and $p_x(y-x+R(d(x, y)))$ is contained in B_x.

From Lemma 2, $(f_x|U_x)^{-1}$ is continuously differentiable on V_x. f_x is continuously differentiable on $G-x$, so the function n_x defined on $R(d(x, y))$ by $n_x(w)=f_x(w+y-x)$ is continuously differentiable.

If w is in $R(d(x, y))\cap U_y$ then

$$\begin{aligned} m_x(m_y^{-1}(w)) &= m_x(g_y^{-1}(w) + y) = m_x(p_y(w) + y) \\ &= g_x(p(w + y) - x) = (p_x \mid U_x)^{-1}(p_x(w + y - x)) \\ &= [(p_x \mid U_x)^{-1} \circ (p'(x) \mid B_x)^{-1} \circ p'(x) \circ p_x](w + y - x) \\ &= (f_x \mid U_x)^{-1} \circ (n_x \mid U_y \cap R(d(x, y)))(w). \end{aligned}$$

This is the composition of differentiable functions so we are done.

LEMMA 5. *If each of x and y is in $p(G)$ then Y_x is linearly homeomorphic to Y_y.*

PROOF. First suppose each of A and B is a continuous linear idempotent mapping from X into X and $\|A-B\|<1$. If I denotes the identity mapping on X then $I+A-B$ is invertible and hence $X=(I+A-B)(X)=(I-B)(X)+A(X)$. Thus $B(X)=B(I-B)(X)+B(A(X))=B(A(X))$ and $(B|A(X))$ is onto $B(X)$.

Suppose x is in $A(X)$. $\|x\|-\|B(x)\|\leqq\|A(x)-B(x)\|\leqq\|A-B\|\,\|x\|$ so $\|B(x)\|\geqq(1-\|A-B\|)\|x\|$. Thus $(B|A(X))$ is one-to-one onto $B(X)$ and, by the closed graph theorem, is a linear homeomorphism.

For each z in $p(G)$ choose D_z to be a neighborhood of z in $p(G)$ so that if w is in D_z then $\|p'(w)-p'(z)\|<\frac{1}{2}$.

Let $A_1=D_x$ and choose inductively $A_n=\bigcup\{D_w : D_w \text{ intersects } A_{n-1}\}$ for $n=2, 3, \cdots$.

Let $A_\infty=\bigcup_1^\infty A_i$. A_∞ is open in $p(G)$. Suppose $\{z_n\}$ is a sequence in A_∞ and $\{z_n\}$ converges to z. Applying p to the sequence and its limit we have: $p(z)=\lim p(z_n)=\lim z_n=z$. Hence z is in $p(G)$. (This argument shows $p(G)$ is closed.) There is an n such that z_n is in D_z. z_n is in A_m for some m so z is in A_{m+1}. Thus A_∞ is closed in $p(G)$ and hence is $p(G)$.

y is in A_n for some n and hence Y_y is linearly homeomorphic to Y_x.

Choose z in $p(G)$ and for each y in $p(G)$ let T_y be a linear homeomorphism from Y_y onto Y_z.

Let $Y=Y_z$ and $h_x=T_x \circ m_x$ for each x in $p(G)$. Since $p'(z)$ is an idempotent, Y_z is the kernel of $I-p'(z)$ and is therefore closed. Moreover, the kernel of $p'(z)$ is a closed complement to Y_z in X, so we satisfy the conditions for a submanifold to be modeled on Y. The closed subspace Y and set of homeomorphisms $\{h_x : x \text{ is in } p(G)\}$ satisfy the conclusion of Theorem 1.

The next theorem uses Theorem 1 to obtain some information on idempotents in differentiable semigroups.

THEOREM 2. *Suppose X is a Banach space, D is an open set of X containing* 0, *and V is a continuously differentiable associative function from $D\times D$ into X satisfying $V(0,0)=0$. Then the set of elements x of D satisfying $V(x,0)=V(0,x)=x$ is locally topologically isomorphic to a local Lie group.*

PROOF. Define p contained in $D\times X$ by $p=\{(x, V(0, V(x,0))) : V(x,0) \text{ is in } D\}$. p is continuously differentiable on a neighborhood of 0 and $p(p(x))=p(x)$ if $p(x)$ is in D.

As before, $p'(0)$ is an idempotent continuous linear transformation from X into X. Let $Y=p'(0)(X)$. From the proofs of Lemmas 2 and 3 there are open sets U and V of Y containing 0 and an open set E of the image of p containing 0 such that $(p'(0) \circ (p|U))^{-1}$ is a continuously differentiable homeomorphism from V onto U and $(p|U)$ is a homeomorphism onto E.

Let $g=(p|U)^{-1}$ and let $A=\{(x,y) \text{ in } U\times U : V(g^{-1}(x), g^{-1}(y)) \text{ is in } \mathrm{dom}(g)\}$. A is open in $Y\times Y$. Define W on A by $W(x,y)=g(V(g^{-1}(x), g^{-1}(y)))$.

$$\begin{aligned} W(x,y) &= [(p \mid U)^{-1} \circ (p'(0) \mid E)^{-1}] \circ p'(0) \circ V \circ (p \times p)(x,y) \\ &= (p'(0) \circ (p \mid U))^{-1} \circ p'(0) \circ V \circ (p \times p)(x,y) \end{aligned}$$

if (x,y) is in A. Thus W is continuously differentiable on a neighborhood of $(0,0)$ in $Y\times Y$ and $W(x,0)=x=W(0,x)$ for each appropriate x in Y. Hence, from [**1**] or in another way from [**3**], $(Y, W, 0)$ is a local Lie group. g^{-1} is the topological isomorphism required for the conclusion of Theorem 2.

REFERENCES

1. G. Birkhoff, *Analytical groups*, Trans. Amer. Math. Soc. **43** (1938), 61–101.

2. J. Dieudonné, *Fondements de l'analyse moderne*, Pure and Appl. Math., vol. 10, Academic Press, New York and London, 1960. MR **22** #11074.

3. J. P. Holmes, *Differentiable power associative groupoids*, Pacific J. Math. **41** (1972), 319–394.

4. S. B. Nadler, Jr., *A characterization of the differentiable submanifolds of* R^n, Proc. Amer. Math. Soc. **17** (1966), 1350–1352. MR **34** #3586.

DEPARTMENT OF MATHEMATICS, AUBURN UNIVERSITY, AUBURN, ALABAMA 36830