LOCALIZATION AT INJECTIVES IN COMPLETE CATEGORIES

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Abstract. We consider a complete category \( \mathcal{A} \). For each object \( I \) of \( \mathcal{A} \) we define a functor \( Q: \mathcal{A} \rightarrow \mathcal{A} \) and obtain a necessary and sufficient condition on \( I \) for \( Q \), after restricting its codomain, to become a reflector of \( \mathcal{A} \) onto the limit closure of \( I \). In particular, this condition is satisfied if \( I \) is injective in \( \mathcal{A} \) with regard to equalizers. Among the special cases of such reflectors are: the reflector onto torsion-free divisible objects associated to an injective \( I \) in \( \text{Mod} \, R \); the Samuel compactification of a uniform space; the Stone-Čech compactification.

We give a second description of \( Q \) in terms of a triple on sets. If \( I \) is injective and the functor \( Q \) is equivalent to the identity then, under a few extra conditions on \( \mathcal{A} \), \( \mathcal{A}^{op} \) is triplable over sets with regard to the functor taking \( A \) to \( \mathcal{A}(A, I) \).

We recall some notation and definitions. We write \( \mathcal{A}(A, B) \) or just \( (A, B) \) for the set of all maps from \( A \) to \( B \) in \( \mathcal{A} \). \( \mathcal{S} \) denotes the category of sets. The limit closure of an object \( I \) of \( \mathcal{A} \) is the smallest full replete subcategory of \( \mathcal{A} \) closed under limits and containing \( I \). \( I \) is injective with regard to the map \( f: A \rightarrow B \) if \( \mathcal{A}(f, I): \mathcal{A}(B, I) \rightarrow \mathcal{A}(A, I) \) is a surjection. We call \( I \) injective in \( \mathcal{A} \) if it is injective with regard to all regular monomorphisms in \( \mathcal{A} \). A regular monomorphism is a map which happens to be an equalizer.

The object \( I \) determines functors

\[
\mathcal{A} \xrightarrow{(-, I)} \mathcal{S}^{op} \xrightarrow{I(-)} \mathcal{A},
\]

where \((-, I)\) is a left adjoint of \( I(-)\), in view of the natural isomorphism

\[
\mathcal{S}^{op}(\mathcal{A}(A, I), X) = \mathcal{S}(X, \mathcal{A}(A, I)) \cong \mathcal{A}(A, I^X).
\]

Thus the composition \( S = I(-, I) \) is part of a triple (standard construction) \( (\mathcal{S}, \eta, \mu) \) on \( \mathcal{A} \) (see [9]). For future reference we describe \( \eta(A): A \rightarrow S(A) \) and also \( S(f) \) for any map \( f: A \rightarrow B \).

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Let $u^*$ denote the map $A \to I^X$ corresponding to $u:X \to (A, I)$, defined by the formula $\forall x \in X \pi_x u^* = u(x)$, where $\pi_x$ is the canonical projection $I^X \to I$. Then $\eta(A) = 1^*_A(A, I)$, that is,

\[(*) \quad \forall v \in \eta(A, I) \pi_v \eta(A) = g.\]

The map $S(f) = I^{f, I}: I^{A, I} \to I^{B, I}$ is given thus

\[(**) \quad \forall v \in \eta(A, B) \forall w \in \eta(B, I) \pi_w S(f) = \pi_{hf}.\]

Following Fakir [2], we define the functor $Q: \mathcal{A} \to \mathcal{A}$ as the equalizer

\[Q \xrightarrow{\kappa} S \xrightarrow{\eta_S} S^2.\]

Fakir showed that $Q$ is part of a triple $(Q, \eta_1, \mu_1)$ and that, if $S(\kappa(A))$ is mono for each object $A$, then $Q$ is idempotent.

Let $\text{Fix } Q$ be the full subcategory of $\mathcal{A}$ consisting of all those objects $A$ for which $\eta_1(A): A \to Q(A)$ is an isomorphism. Since $Q(A)$ is only defined up to isomorphism, we can assume that each such $\eta_1(A)$ is the identity map of $A$. Then $Q$ is idempotent if and only if, by restriction of the codomain to the image, it induces a reflector $\mathcal{A} \to \text{Fix } Q$. The reflection map from $A$ into $\text{Fix } Q$ is then $\eta_1(A): A \to Q(A)$, which is defined by the condition $\eta_1(A) \eta_1(A) = \eta(A)$.

Before stating our main result, we require two lemmas.

**Lemma 1.** $\kappa(A): Q(A) \to S(A)$ is the joint equalizer of all pairs of maps $S(A) \xrightarrow{\eta_1} I$ which coequalize $\eta(A)(A) \to S(A)$.

**Proof.** Consider any map $u:S(A) \to I$. Then, by $(*)$, $\pi_u \eta S(\dot{A}) = u$, and by $(**)$, $\pi_u S(\eta(A)) = \pi_{u \eta_1(A)}$. Thus $\kappa(A)$ equalizes all pairs of maps $(u, \pi_{u \eta_1(A)})$. Now let $v:S(A) \to I$ be such that $u \eta_1(A) = v \eta_1(A)$. Then $\kappa(A)$ equalizes $(u, v)$, since

\[u \kappa(A) = \pi_{u \eta_1(A)} \kappa(A) = \pi_{u \eta_1(A)} \kappa(A) = v \kappa(A).\]

Conversely, any map which equalizes all $(u, v)$ such that $u \eta_1(A) = v \eta_1(A)$ equalizes $(u, \pi_{u \eta_1(A)})$ in particular, since $\pi_{u \eta_1(A)} \eta(A) = u \eta_1(A)$ by $(*)$. Hence it equalizes $\eta S(A)$ and $S \eta(A)$.

**Lemma 2.** $I \in \text{Fix } Q$.

**Proof.** By $(*)$, $\pi_I \eta(I) = 1_{I}$, hence $\eta(I)$ is the equalizer of the pair of maps $(\eta(I) \pi_1, 1_{S(I)})$. Thus $\eta(I)$ is the joint equalizer of all pairs of maps $I^{f, I} \to I^{(I, I)}$ which coequalize $\eta(I)$, and therefore the joint equalizer of all pairs of maps $I^{f, I} \to I$ which coequalize $\eta(I)$. In view of Lemma 1,
\( \kappa(I) \) is this same equalizer. Hence the unique map \( \eta_1(I): I \to Q(I) \) such that 
\( \kappa(I) \eta_1(I) = \eta(I) \) is an isomorphism, and so \( I \) is in \( \text{Fix } Q. \)

**Theorem.** The following statements are equivalent.

(a) \( I \) is injective with regard to \( \kappa(A) \) for each object \( A \) in \( \mathcal{A} \).

(b) \( Q \) is idempotent, i.e. becomes a reflector \( \mathcal{A} \to \text{Fix } Q \) when its co-
domain is restricted to its image.

(c) \( \text{Fix } Q \) is the limit closure of \( I \).

**Proof.** (b) and (c) are clearly equivalent, since

\[ I \in \text{Fix } Q \subset \text{image of } Q \subset \text{limit closure of } I \]

and a reflective subcategory is limit closed.

(a) \( \Rightarrow \) (b). We note that if \( I \) is injective with regard to a map \( f \) then
\( (f, I) \) is a surjection, that is, a regular mono of \( \mathcal{A}^{\text{op}}. \) Now \( I \to - \) being a right adjoint, preserves limits, so \( S(f) = I(f \cdot I) \) is a regular mono of \( \mathcal{A}. \)
Thus it follows from condition (a) that \( S(\kappa(A)) \) is mono for each \( A. \)
Fakir’s result (Proposition 3 of \[2\]) then shows that \( Q \) is idempotent.

(b) \( \Rightarrow \) (a). We have \( A \to \eta_1(A)Q(A) \to \kappa(A)S(A) \) with \( \kappa(A) \eta_1(A) = \eta(A). \)
Now every map \( u: A \to I \) can be extended to \( \pi_u : S(A) \to I, \) since \( \pi_u \eta(A) = u. \)
Given any map \( v: Q(A) \to I, \) we let \( u = v \eta_1(A). \) Then \( \pi_u \kappa(A) \eta_1(A) = u = v \eta_1(A). \) Since \( \eta_1(A) \) is a reflection map of \( A \) into \( \text{Fix } Q, \) it follows from
Lemma 2 that \( \pi_u \kappa(A) = v. \) Thus \( I \) is injective with regard to \( \kappa(A). \)
This completes the proof of the theorem.

**Definition.** We shall call the object \( I \) of \( \mathcal{A} \) \( \kappa \)-injective if it satisfies
the equivalent conditions of the theorem. We call \( Q \) the localization
functor determined by \( I. \)

**Corollary.** If \( I \) is injective with regard to all equalizers of pairs of
maps \( I^X \rightrightarrows I^Y \) then \( Q \) is (after restricting its codomain) a reflector onto the
limit closure of \( I, \) which is \( \text{Fix } Q. \) If \( I \) is injective in \( \mathcal{A} \) then \( Q \) takes regular
monos of \( \mathcal{A} \) to regular monos of \( \text{Fix } Q. \)

**Proof.** The first statement is obvious. As for the second, let \( I \) be
injective in \( \mathcal{A} \) and \( f: A \to B \) a regular mono of \( \mathcal{A}. \) We already know that
\( S(f) \) is a regular mono of \( \mathcal{A} \) which lies in \( \text{Fix } Q. \) Since \( \text{Fix } Q \) is the limit
closure of the injective \( I, \) Theorem 1 of \[11\] shows that \( S(f) \) is a regular
mono of \( \text{Fix } Q. \) Now \( \kappa(A):Q(A) \to S(A) \) is also a regular mono of \( \text{Fix } Q \)
and it is easily seen that the composition of two regular monos of \( \text{Fix } Q \)
is a regular mono of \( \text{Fix } Q \) (since an object \( A \) is in \( \text{Fix } Q \) iff there is an
equalizer diagram \( A \rightrightarrows I^p \to I^q \)). Thus \( \kappa(B)Q(f) = S(f) \kappa(A) \) is a regular

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\[1\] The referee has pointed out that Lemma 2 can also be proved directly by observing that \( \pi_1 \) and \( S(\pi_1) \) make \( I \to S(I) \to S^2(I) \) into a split equalizer diagram.
mono of \( \text{Fix } Q \). Since \( \kappa(B) \) is mono in \( \text{Fix } Q \), \( Q(f) \) is also a regular mono of \( \text{Fix } Q \).

We remark that if \( \mathcal{A} \) is well powered then, by the special adjoint functor theorem, the limit closure of any object \( I \) is a reflective subcategory of \( \mathcal{A} \).

We shall now consider a number of examples. Further examples in the categories of partially ordered sets, lattices, etc. are being studied by B. Ballinger.

**Example 1.** Take \( \mathcal{A} = \text{Mod } R \), where \( R \) is an associative ring with unity, and let \( I \) be any injective right \( R \)-module. We claim that \( Q(A) \) is then the usual localization of \( A \), also called the *module of quotients* of \( A \), with respect to the torsion theory determined by \( I \) (see [7]).

To prove this, let us provisionally denote the module of quotients by \( Q'(A) \). Then \( Q'(A) \) is divisible (with respect to \( I \)), in the terminology of [7] and may be regarded as a submodule of \( \text{S}(A) \), since the kernel of \( \eta(A) \) is the torsion submodule of \( A \). Now \( \text{S}(A) \), being a product of copies of \( I \), is torsion-free (with respect to \( I \)), hence \( \text{S}(A)/Q'(A) \) is torsion-free. Since \( Q'(A)/\text{Im } \eta(A) \) is torsion, it is the torsion submodule of \( \text{S}(A)/\text{Im } \eta(A) \).

Thus \( s \in Q'(A) \) if and only if

\[
\forall u: S(A) \to I(u\eta(A) = 0 \Rightarrow u(s) = 0),
\]

that is,

\[
\forall v, w: S(A) \to I(v\eta(A) = w\eta(A) \Rightarrow v(s) = w(s)).
\]

Therefore \( Q'(A) \) is the joint equalizer of all pairs of maps which coequalize \( \eta(A) \). Hence, by Lemma 1, \( Q'(A) = Q(A) \), as was to be proved.

**Example 2.** Take \( \mathcal{A} \) to be the category of topological spaces and let \( I \) be the unit interval \([0, 1]\). \( I \) is not injective, but it is \( \kappa \)-injective. It is easily seen that \( Q(A) \) is the closure in \( S(A) \) of the image of \( \eta(A) \). Thus the construction of \( Q(A) \) is the familiar construction of the Stone-Čech compactification \( \beta(A) \), as described by Čech [1]. Condition (a) in this example is simply a special case of Tietze's theorem, since \( S(A) = I(A, I) \) is normal and \( Q(A) \) is a closed subspace.

Of course, there is no reason for \( \beta \) to preserve regular monos, as \( I \) is not injective (with regard to all regular monos) in \( \mathcal{A} \).²

**Example 3.** Take \( \mathcal{A} \) to be the category of uniform spaces (not necessarily Hausdorff) and let \( I = [0, 1] \). Regular monomorphisms in \( \mathcal{A} \) are easily seen to be the same as subspace inclusions. It is known that \( I \) is injective with regard to subspace inclusions (see [6]).

The reflector \( Q \) is the Samuel compactification (see [12] or [5]), that is, \( \text{Fix } Q \) consists of all compact Hausdorff uniform spaces.

² George Reynolds has observed that one can similarly obtain the real compactification of \( A \) by taking \( I \) to be the real line.
To see this, we note that $Q(A)$ is easily shown to be the closure in $S(A)$ of the image of $\eta(A)$. Now $S(A) = I^{(A,I)}$ is compact, and therefore so is the closed subspace $Q(A)$. (We observe that the forgetful functor from uniform to topological spaces preserves products.) Conversely, assume that $A$ is compact Hausdorff, and recall that all continuous mappings from a compact Hausdorff uniform space are uniformly continuous. Then $Q(A)$ is the Stone-Čech compactification of $A$, and hence $\eta_1(A)$ is an isomorphism.

There are many other injectives in $\mathcal{A}$. For example, if $M$ is any metrizable uniform space then $(M, I)$ with the obvious metric is injective $[5]$. It is not clear what the associated category Fix $Q$ is. It certainly contains only complete spaces. It seems unlikely that there is any $M$ for which Fix $Q$ contains all complete spaces, that is, such that $Q(A)$ is the Hausdorff completion of $A$.

Example 4. Let $\mathcal{A}$ be the category of presheaves on a small category $\mathcal{X}$, that is, the category of all functors $\mathcal{X}^{\text{op}} \to \mathcal{S}$. Suppose that $\mathcal{X}$ is equipped with a Grothendieck topology, then one can construct a huge injective $I$ whose limit closure is the category of sheaves for the given topology. For any presheaf $A$, $Q(A)$ is then the associated sheaf.

The story is somewhat different when $\mathcal{A}$ is an elementary topos in the sense of Lawvere and Tierney. Since $\mathcal{A}$ is not necessarily complete, our construction of the triple $S$ does not work. However, an analogous construction does work, and we shall sketch it briefly.

Let $I$ be any injective in $\mathcal{A}$. Since $\mathcal{A}$ is a cartesian closed category, we can obtain a triple $S$ on $\mathcal{A}$ from the selfadjoint functor $\mathcal{A} \to \mathcal{A}^{\text{op}}$ taking $A$ to $I^A$, with $S(A) = I^A$. Fakir's construction applied to $S$ then gives an idempotent triple $Q$. However, Fix $Q$ is not the limit closure of $I$; it must also be closed under internal powers. It turns out that $Q$ preserves all finite limits.

The work of Lawvere and Tierney suggests which injectives $I$ one should single out for consideration: Let $j$ be any Heyting endomorphism of the subobject classifier $\Omega$, then take $I = \Omega_j$, the retract of $\Omega$ determined by $j$. Fix $Q$ will then be the category of $j$-sheaves. (For definitions see the discussion of the work of Lawvere and Tierney in $[3]$.)

We plan to elaborate the details of this example in a sequel to the present paper.

We shall give another interpretation of the localization functor $Q$.

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3 Indeed, George Reynolds has observed that the completion functor in uniform spaces cannot be obtained by our method from a single uniform space $I$, because any such space has a cardinal number associated with it which is preserved by products and subspace formation, namely the smallest infinite cardinal such that every uniform cover has a refinement less than it.
with the help of the Eilenberg-Moore category of a triple. It will be con-
venient here to replace the complete category \( \mathcal{A} \) by its opposite, the co-
complete category \( \mathcal{A}^{\text{op}} \). We consider an object \( P \) of \( \mathcal{B} \) which is
\( \kappa \)-projective, that is, \( \kappa \)-injective as an object of \( \mathcal{A} \).

The functor \( U = (P, -): \mathcal{B} \to \mathcal{S} \) has a left adjoint \( F \), where \( F(X) = \sum_{x \in X} P \), let us say with adjunction \( \eta: \text{id} \to UF \) and coadjunction \( \epsilon: FU \to \text{id} \). The triple \( (UF, \eta, U\epsilon F) \) on \( \mathcal{S} \) gives rise to the Eilenberg-Moore category \( \mathcal{S}^{UF} \), whose objects are certain pairs \((X, \xi)\), where \( \xi: UF(X) \to X \) in \( \mathcal{S} \).

One studies the so-called comparison functor \( K: \mathcal{B} \to \mathcal{S}^{UF} \) given by
\[
K(B) = (U(B), U\xi(B)), \quad K(b) = U(b).
\]
This has a left adjoint \( M \) (see [9, p. 151, Exercise 5]), where \( M(X, \xi) \) is the coequalizing object of the pair
\[
FUF(X) \xrightarrow{F(\xi)} F(X).
\]

A simple calculation shows the following:

**Proposition 1.** If \( K \) is the comparison functor of \((\mathcal{B}, U)\), \( M \) its left
adjoint, and \( Q \) the localization functor on \( \mathcal{B}^{\text{op}} \), then \( MK = Q \).

We may also call \( Q \) the colocalization functor on \( \mathcal{B} \).

We consider an interesting special case.

**Example 5.** Take \( \mathcal{B} = \text{Mod} R \) and let \( P \) be any finitely generated
\( \kappa \)-projective right \( R \)-module. Let \( E \) be the ring \((P, P)\), then \((P, -)\) may
be considered as a functor \( \mathcal{B} \to \text{Mod} E \) with left adjoint \((-) \otimes_{E} P \). Since \( P \)
is finitely generated, \((P, -)\) takes sums in \( \mathcal{B} \) to sums in \( \text{Mod} E \). Let \( U_{E} \)
denote the forgetful functor \( \text{Mod} E \to \mathcal{S} \). Then the triple \( UF \) is given by
\[
UF(X) = U_{E}\left(P, \sum_{X} P\right) \cong U_{E}\left(\sum_{X} E\right).
\]
Now this is the triple associated to \( \text{Mod} E \), so \( \mathcal{S}^{UF} = \text{Mod} E \). The functor \( K \) is clearly \((P, -)\), hence \( M = ( - ) \otimes_{E} P \), and we have
\[
Q(B) = (P, B) \otimes_{E} P.
\]
This formula actually holds in a more general situation. Let \( \mathcal{B} \) be any
cocomplete abelian category and \( P \) any \( \kappa \)-projective object which is small
in the sense that \((P, -): \mathcal{B} \to \text{Mod} E \) preserves sums.

The smallness of \( P \) will follow easily if it is assumed to be finitely
generated in the sense of [4], that is, that \((P, -)\) preserves directed colimits
of monomorphisms. In this definition it makes no difference whether \((P, -)\) is considered as a functor into \( \text{Mod} E \) or into \( \mathcal{S} \).

\[\text{[4]}\]

\[\text{We are indebted to the referee for criticizing the original discussion of this example, in which the condition that } P \text{ be finitely generated had been overlooked.}\]
Proposition 2. Let $Q$ be the colocalization functor determined by the $\kappa$-projective object $P$ of the cocomplete category $\mathcal{B}$. Then the following conditions are equivalent.

1. $Q$ is canonically isomorphic to the identity, that is, $\eta_1$ is a natural isomorphism.
2. $P$ is a regular generator in the sense that for each object $B$ of $\mathcal{B}$ there is a regular epi from some multiple of $P$ to $B$.
3. For each object $B$ of $\mathcal{B}$ there is a coequalizer diagram $mP \xrightarrow{n} nP \rightarrow B$, for some cardinal numbers $m$ and $n$.
4. $\mathcal{B}$ is the colimit closure of $P$.

Proof. The implication $(4) \Rightarrow (1)$ is an immediate consequence of our theorem. For, since $P$ is $\kappa$-projective, the theorem tells us that the colimit of $P$ is $\text{Fix } Q$, and by (4) this is $\mathcal{B}$, so that (1) holds.

The implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ hold even without the assumption that $P$ is $\kappa$-projective. The first and last implications are clear; we shall prove that $(2) \Rightarrow (3)$. Assume $(2)$ and let $B$ be any object of $\mathcal{B}$. Then there is a coequalizer diagram $B' \xrightarrow{n} nP \rightarrow B$ and a regular epi $mP \rightarrow B'$, hence a coequalizer diagram $mP \xrightarrow{n} nP \rightarrow B$.

Clearly $(1)$ implies that $P$ is $\kappa$-projective. Hence $(1)$ asserts that $P$ is a $\kappa$-projective regular generator.

We present the following variant of Linton's theorem (see [8, p. 88]).

Proposition 3. Let $P$ be an object of the category $\mathcal{B}$ and $U=\mathcal{B}(P, -)$, then $(\mathcal{B}, U)$ is varietal (triplable) if and only if

1. $\mathcal{B}$ is cocomplete and has kernel pairs,
2. $P$ is a projective regular generator,
3. every equivalence relation in $\mathcal{B}$ is a kernel pair.

A pair of maps $R \xrightarrow{A}$ in $\mathcal{B}$ is here called an equivalence relation if $\mathcal{B}(B, R) \xrightarrow{A} \mathcal{B}(B, A)$ is an equivalence relation in $\mathcal{S}$ for every object $B$ of $\mathcal{B}$.

Proof. Necessity of conditions. (1) and (3) are well-known properties of varietal categories, and (2) follows from Proposition 1.

Sufficiency of conditions. We verify Linton's conditions FIT ([8, p. 88]). We know that $U$ has a left adjoint and that $(FIT)_0$ holds, that is, $\mathcal{B}$ has kernel pairs and coequalizers. $(FIT)_1$ says that $\pi: A \rightarrow B$ in $\mathcal{B}$ is a regular epi if and only if $U(\pi)$ is a surjection.

Since $P$ is projective, $U$ takes regular epis to surjections. Conversely, if $U(\pi)$ is a surjection, then $FU(\pi)$ is a regular epi, since $F$ preserves

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Both M. Barr and C. Mulvey have informed us that they have obtained essentially the same version of Linton's theorem.
limits. Now consider the following commutative diagram:

\[
\begin{array}{ccc}
FU(A) & \xrightarrow{FU(\pi)} & FU(B) \\
\varepsilon(A) \downarrow & & \downarrow \varepsilon(B) \\
A & \xrightarrow{\pi} & B
\end{array}
\]

Since \( \mathcal{C} \cong \text{id} \) canonically, \( \varepsilon(A) \) and \( \varepsilon(B) \) are regular epis. Now the composition \( \varepsilon(B)FU(\pi) \) of two regular epis is a regular epi, since \( \mathcal{B} \) is the limit closure of the projective \( P \) (see [11, Lemma 2.2]). Since \( \pi \varepsilon(A) \) is a regular epi and \( \varepsilon(A) \) is epi, \( \pi \) is a regular epi.

\((FIT)_2\) says that \( p_1, p_2: R \rightarrow A \) is a kernel pair in \( \mathcal{B} \) if and only if \( U(p_1) \), \( U(p_2): U(R) \rightarrow U(A) \) is a kernel pair in \( \mathcal{S} \). The "only if" part is obvious, since \( U \) is representable. Consider the full subcategory \( \mathcal{C} \) of \( \mathcal{B} \) consisting of those objects \( B \) for which the representable functor \( (B, -) \) takes \( (p_1, p_2) \) into a kernel pair. \( \mathcal{C} \) contains \( P \) and is obviously replete. It is easily seen to be closed under colimits, and since \( \mathcal{B} \) is the colimit closure of \( P \), \( \mathcal{C} = \mathcal{B} \).

**Corollary 1.** Let \( \mathcal{A} \) be a complete Abelian category with an injective cogenerator \( I \). Then \( \mathcal{A}^{\text{op}} \) is varietal with respect to the functor \( \mathcal{A}(-, I) \).

**Proof.** \( \mathcal{B} = \mathcal{A}^{\text{op}} \) satisfies the conditions of Proposition 3 with \( P = I \). Indeed, conditions (1) and (2) are obvious. To prove (3) we observe that in \( \text{Mod } R \) every equivalence relation is a kernel pair. By Mitchell's embedding theorem, the same is true in any Abelian category, hence in the opposite of an Abelian category.

**Example 6.** The opposite of any Grothendieck category \( \mathcal{A} \) is varietal, as \( \mathcal{A} \) contains an injective cogenerator. Oberst [10] has also described \( \mathcal{A}^{\text{op}} \) as a concrete category with the forgetful functor \( \mathcal{A}(-, I) \). However, the structure he defines involves topology and is not obviously varietal.

**Corollary 2.** Let \( Q \) be the colocalization functor associated with the projective object \( P \) of the cocomplete category \( \mathcal{B} \). Assume \( \mathcal{B} \) has kernel pairs and all equivalence relations in the category \( \text{Fix } Q \) are kernel pairs. Then \( \text{Fix } Q \) is varietal with respect to the functor \( (P, -): \text{Fix } Q \rightarrow \mathcal{S} \).

**Proof.** \( \text{Fix } Q \) satisfies conditions (1) to (3) of Proposition 3.

Example 6 could also have been deduced from Corollary 2 by means of the Gabriel-Popescu theorem.

**Example 7.** Let \( \mathcal{A} \) be the category of all set-valued sheaves with respect to a Grothendieck topology. Then \( \mathcal{A}^{\text{op}} \) is varietal.

A proof will be given in a sequel to this paper.
References


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