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ON THE DERIVATIVE OF A POLYNOMIAL

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ABSTRACT. If $p(z)=\sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having all its zeros in $|z|\leqq K\leqq 1$, then it is known that $\max_{|z|=1}|p'(z)|\geqq (n/(1+K))\max_{|z|=1}|p(z)|$. In this paper we consider the case when $K>1$ and obtain a sharp result.

1. The following result is due to Turán [4].

THEOREM A. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z|\leqq 1$ then*

$$\max_{|z|=1}|p'(z)|\geqq \frac{n}{2}\max_{|z|=1}|p(z)|. \tag{1}$$

The result is sharp and equality holds in (1) *if all the zeros of $p(z)$ lie on $|z|=1$.*

More generally if $p(z)=a_n\prod_{v=1}^{n}(z-z_v)$ is a polynomial of degree n having all its zeros in $|z|\leqq K\leqq 1$, then

$$\left|\frac{p'(e^{i\theta})}{p(e^{i\theta})}\right| \geqq \operatorname{Re}\left(e^{i\theta}\frac{p'(e^{i\theta})}{p(e^{i\theta})}\right) = \sum_{v=1}^{n}\operatorname{Re}\left(\frac{e^{i\theta}}{e^{i\theta}-z_v}\right) \geqq \sum_{v=1}^{n}\frac{1}{1+K}$$

i.e.

$$|p'(e^{i\theta})| \geqq (n/(1+K))\,|p(e^{i\theta})|, \qquad \theta \text{ real.}$$

Choosing θ such that $|p(e^{i\theta})|=\max_{|z|=1}|p(z)|$, we get

$$\max_{|z|=1}|p'(z)|\geqq \frac{n}{1+K}\max_{|z|=1}|p(z)|. \tag{2}$$

In (2) equality holds for the polynomial $p(z)=((z+K)/(1+K))^n$.

The above argument does not hold for $K>1$ for then $\operatorname{Re}(e^{i\theta}/(e^{i\theta}-z_v))$ may not be $\geqq 1/(1+K)$.

Another proof of (2) is given in [2] where it is deduced by applying the following result (for another proof see [1, Theorem C, p. 503]) to the polynomial $z^n p(1/z)$.

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THEOREM B. *If $p(z)$ is a polynomial of degree n, with $|p(z)|\leqq 1$ on $|z|\leqq 1$ and $p(z)$ has no zero in the disk $|z|<K$, $K\geqq 1$, then for $|z|\leqq 1$,*

$$|p'(z)| \leqq n/(1+K). \tag{3}$$

The result is best possible and equality in (3) *holds for* $p(z)=((z+K)/(1+K))^n$.

Thus again the question as to what happens to (2) if $K>1$ remains unanswered. We settle the case $K>1$ by proving the following theorem.

THEOREM. *If $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n with $\max_{|z|=1}|p(z)|=1$ and $p(z)$ has all its zeros in the disk $|z|\leqq K$, $K\geqq 1$, then*

$$\max_{|z|=1}|p'(z)| \geqq \frac{n}{1+K^n}. \tag{4}$$

The result is best possible with equality for the polynomial $p(z)=(z^n+K^n)/(1+K^n)$.

For $K>1$ the extremal polynomial turns out to be of the form $(z^n+K^n)/(1+K^n)$ whereas for $K<1$ it has the form $((z+K)/(1+K))^n$. Thus 1 is a critical value of K for the problem under consideration and one should not expect the same kind of reasoning to work both for $K<1$ and for $K>1$.

2. For the proof of the theorem, we need the following lemmas.

LEMMA 1. *If $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having all its zeros in the disk $|z|\leqq K$, $K\geqq 1$ then for $0\leqq\theta<2\pi$, $|p'(K^2e^{i\theta})|\geqq K^{n-2}|q'(e^{i\theta})|$, when $q(z)=z^n\overline{(p(1/\bar{z}))}$.*

PROOF OF LEMMA 1. The polynomial $P_1(z)\equiv p(Kz)$ has all its zeros in the unit disk $|z|\leqq 1$ and so the polynomial $Q_1(z)=z^n\overline{(P_1(1/\bar{z}))}\equiv z^n\overline{(p(K/\bar{z}))}\equiv K^n q(z/K)$ has all its zeros in $|z|\geqq 1$. Since $|P_1(z)|=|Q_1(z)|$ on $|z|=1$, it follows that $|Q_1(z)|\leqq|P_1(z)|$ for $|z|\geqq 1$. Hence $Q_1(z)-\lambda P_1(z)$ has all its zeros in $|z|<1$ if $|\lambda|>1$. It then follows by the Gauss-Lucas theorem that all the zeros of the polynomial $Q_1'(z)-\lambda P_1'(z)$ also lie in $|z|<1$, which implies that $|Q_1'(z)|\leqq|P_1'(z)|$ for $|z|\geqq 1$. In particular $K^{n-1}|q'(e^{i\theta})|\leqq K|p'(K^2e^{i\theta})|$ and the lemma follows.

LEMMA 2. *If $p(z)$ is a polynomial of degree n having all its zeros in the disk $|z|\leqq K$, $K\geqq 1$, then*

$$\max_{|z|=1}|q'(z)| \leqq K^n \max_{|z|=1}|p'(z)|, \tag{5}$$

where $q(z)$ is as defined in Lemma 1.

PROOF OF LEMMA 2. By Lemma 1,

$$\max_{|z|=1} |q'(z)| \leqq \frac{1}{K^{n-2}} \max_{|z|=K^2} |p'(z)|.$$

But if $f(z)$ is a polynomial of degree n such that $|f(z)| \leqq M$ on $|z|=1$ then $|f(z)| \leqq MR^n$ for $|z|=R>1$ (see [**3**, Problem 269, page 137]). Hence $\max_{|z|=K^2} |p'(z)| \leqq K^{2n-2} \max_{|z|=1} |p'(z)|$, from which the lemma follows.

LEMMA 3. *If $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n,*

$$\max_{|z|=1} |p(z)| = 1,$$

then on $|z|=1$

$$|p'(z)| + |q'(z)| \leqq n, \tag{6}$$

where $q(z)$ is as defined in Lemma 1.

This is a special case of a result due to Govil and Rahman [**1**, Lemma 10].

LEMMA 4. *If $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n,*

$$\max_{|z|=1} |p(z)| = 1, \qquad p(z) \equiv q(z)$$

where $q(z)$ is as defined in Lemma 1, *then on $|z|=1$,*

$$\max_{|z|=1} |p'(z)| = \frac{n}{2}. \tag{7}$$

PROOF OF LEMMA 4. If $p(z)\equiv q(z)$, it follows by Lemma 3,

$$\max_{|z|=1} |p'(z)| = \max_{|z|=1} |q'(z)| \leqq \frac{n}{2}.$$

Since on $|z|=1$, $|q'(z)| = |np(z)-zp'(z)|$ we get for $|z|=1$,

$$n/2 \geqq |q'(z)| = |np(z) - zp'(z)| \geqq n\,|p(z)| - |p'(z)|.$$

Choosing z on $|z|=1$ for which $|p(z)|$ becomes maximum, we get $\max_{|z|=1} |p'(z)| \geqq n/2$ and the lemma follows.

3. PROOF OF THE THEOREM. For every ε, $|\varepsilon|=1$, the polynomial $P^*(z)=\frac{1}{2}\{p(z)+\varepsilon q(z)\}$ satisfies $P^*(z)\equiv z^n(P^*(1/\bar{z}))^-$ and $\max_{|z|=1} |P^*(z)| = 1$, hence by Lemma 4

$$\max_{|z|=1} |p'(z) + \varepsilon q'(z)| \geqq n,$$

which implies

$$\max_{|z|=1} |p'(z)| + \max_{|z|=1} |q'(z)| \geqq n.$$

Applying Lemma 2, we get

$$\max_{|z|=1} |p'(z)| + K^n \max_{|z|=1} |p'(z)| \geqq n,$$

and the theorem follows.

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