ON THE DERIVATIVE OF A POLYNOMIAL

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Abstract. If \( p(z) = \sum_{v=0}^{n} a_v z^v \) is a polynomial of degree \( n \) having all its zeros in \( |z| \leq K \leq 1 \), then it is known that \( \max_{|z|=1} |p'(z)| \geq \frac{n}{(1+K)} \max_{|z|=1} |p(z)| \). In this paper we consider the case when \( K > 1 \) and obtain a sharp result.

1. The following result is due to Turán [4].

Theorem A. If \( p(z) \) is a polynomial of degree \( n \) having all its zeros in \( |z| \leq 1 \) then

\[
\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|. 
\]

The result is sharp and equality holds in (1) if all the zeros of \( p(z) \) lie on \( |z|=1 \).

More generally if \( p(z) = a_n \prod_{v=1}^{n} (z - z_v) \) is a polynomial of degree \( n \) having all its zeros in \( |z| \leq K \leq 1 \), then

\[
\left| \frac{p'(e^{i\theta})}{p(e^{i\theta})} \right| \geq \frac{\operatorname{Re} \left( \frac{e^{i\theta} p'(e^{i\theta})}{p(e^{i\theta})} \right)}{\sum_{v=1}^{n} \frac{1}{1+K}} \geq \sum_{v=1}^{n} \frac{1}{1+K} 
\]
i.e.

\[
|p'(e^{i\theta})| \geq \frac{(n/(1+K))}{|p(e^{i\theta})|}, \quad \theta \text{ real.}
\]

Choosing \( \theta \) such that \( |p(e^{i\theta})|=\max_{|z|=1} |p(z)| \), we get

\[
\max_{|z|=1} |p'(z)| \geq \frac{n}{1+K} \max_{|z|=1} |p(z)|. 
\]

In (2) equality holds for the polynomial \( p(z) = ((z+K)/(1+K))^n \).

The above argument does not hold for \( K>1 \) for then \( \operatorname{Re}(e^{i\theta}/(e^{i\theta}-z_v)) \) may not be \( \geq 1/(1+K) \).

Another proof of (2) is given in [2] where it is deduced by applying the following result (for another proof see [1, Theorem C, p. 503]) to the polynomial \( z^n p(1/z) \).

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Theorem B. If \( p(z) \) is a polynomial of degree \( n \), with \(|p(z)| \leq 1\) on \(|z| \leq 1\) and \( p(z) \) has no zero in the disk \(|z| < K\), \( K \geq 1\), then for \(|z| \leq 1\),

\[
|p'(z)| \leq n/(1 + K).
\]

The result is best possible and equality in (3) holds for \( p(z) = ((z+K)/(1+K))^n \).

Thus again the question as to what happens to (2) if \( K > 1 \) remains unanswered. We settle the case \( K > 1 \) by proving the following theorem.

Theorem. If \( p(z) = \sum_{r=0}^{n} a_r z^r \) is a polynomial of degree \( n \) with \( \max_{|z|=1} |p(z)| = 1 \) and \( p(z) \) has all its zeros in the disk \(|z| \leq K\), \( K \geq 1\), then

\[
\max_{|z|=1} |p'(z)| \geq \frac{n}{1 + K^n}.
\]

The result is best possible with equality for the polynomial \( p(z) = (z^n + K^n)/(1 + K^n) \).

For \( K > 1 \) the extremal polynomial turns out to be of the form \((z^n + K^n)/(1 + K^n)\) whereas for \( K < 1 \) it has the form \(((z + K)/(1 + K))^n\). Thus 1 is a critical value of \( K \) for the problem under consideration and one should not expect the same kind of reasoning to work both for \( K < 1 \) and for \( K > 1 \).

2. For the proof of the theorem, we need the following lemmas.

Lemma 1. If \( p(z) = \sum_{r=0}^{n} a_r z^r \) is a polynomial of degree \( n \) having all its zeros in the disk \(|z| \leq K\), \( K \geq 1\) then for \( 0 \leq \theta < 2\pi \), \(|p'(K e^{i\theta})| \geq K^{n-2}|q'(e^{i\theta})|\), where \( q(z) = z^n(p(1/\bar{z}))^{-1} \).

Proof of Lemma 1. The polynomial \( P_1(z) = p(Kz) \) has all its zeros in the unit disk \(|z| \leq 1\) and so the polynomial \( Q_1(z) = z^n(P_1(1/\bar{z}))^{-1} = z^n(p(K/\bar{z}))^{-1} = K^n q(z/K) \) has all its zeros in \(|z| \geq 1\). Since \(|P_1(z)| = |Q_1(z)|\) on \(|z| = 1\), it follows that \(|Q_1(z)| \leq |P_1(z)|\) for \(|z| \geq 1\). Hence \( Q_1(z) - \lambda P_1(z) \) has all its zeros in \(|z| < 1\) if \(|\lambda| > 1\). It then follows by the Gauss-Lucas theorem that all the zeros of the polynomial \( Q_1(z) - \lambda P_1(z) \) also lie in \(|z| < 1\), which implies that \(|Q_1(z)| \leq |P_1(z)|\) for \(|z| \geq 1\). In particular \( K^{n-1}|q'(e^{i\theta})| \leq K|p'(K^2 e^{i\theta})|\) and the lemma follows.

Lemma 2. If \( p(z) \) is a polynomial of degree \( n \) having all its zeros in the disk \(|z| \leq K\), \( K \geq 1\), then

\[
\max_{|z|=1} |q'(z)| \leq K^n \max_{|z|=1} |p'(z)|,
\]

where \( q(z) \) is as defined in Lemma 1.
Proof of Lemma 2. By Lemma 1,

$$\max_{|z|=1} |q'(z)| \leq \frac{1}{K^{n-2}} \max_{|z|=K^2} |p'(z)|.$$  

But if $f(z)$ is a polynomial of degree $n$ such that $|f(z)| \leq M$ on $|z|=1$ then $|f(z)| \leq MR^n$ for $|z|=R>1$ (see [3, Problem 269, page 137]). Hence

$$\max_{|z|=K^2} |p'(z)| \leq K^{2n-2} \max_{|z|=1} |p'(z)|,$$

from which the lemma follows.

Lemma 3. If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree $n$,

$$\max_{|z|=1} |p(z)| = 1,$$

then on $|z|=1$

(6) $$|p'(z)| + |q'(z)| \leq n,$$

where $q(z)$ is as defined in Lemma 1.

This is a special case of a result due to Govil and Rahman [1, Lemma 10].

Lemma 4. If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree $n$,

$$\max_{|z|=1} |p(z)| = 1, \quad p(z) \equiv q(z)$$

where $q(z)$ is as defined in Lemma 1, then on $|z|=1$,

(7) $$\max_{|z|=1} |p'(z)| = \frac{n}{2}.$$  

Proof of Lemma 4. If $p(z) \equiv q(z)$, it follows by Lemma 3,

$$\max_{|z|=1} |p'(z)| = \max_{|z|=1} |q'(z)| \leq \frac{n}{2}.$$  

Since on $|z|=1$, $|q'(z)| = |np(z) - zp'(z)|$ we get for $|z|=1$,

$$n/2 \geq |q'(z)| = |np(z) - zp'(z)| \geq n |p(z)| - |p'(z)|.$$  

Choosing $z$ on $|z|=1$ for which $|p(z)|$ becomes maximum, we get

$$\max_{|z|=1} |p'(z)| \geq n/2$$

and the lemma follows.

3. Proof of the theorem. For every $\varepsilon$, $|\varepsilon|=1$, the polynomial

$P^*(z) = \frac{1}{2} \{p(z) + \varepsilon q(z)\}$ satisfies $P^*(z) \equiv z^n (P^*(1/\bar{z}))$ and $\max_{|z|=1} |P^*(z)| = 1$, hence by Lemma 4

$$\max_{|z|=1} |p'(z) + \varepsilon q'(z)| \geq n.$$
which implies
\[ \max_{|z|=1} |p'(z)| + \max_{|z|=1} |q'(z)| \geq n. \]

Applying Lemma 2, we get
\[ \max_{|z|=1} |p'(z)| + K^n \max_{|z|=1} |p'(z)| \geq n, \]

and the theorem follows.

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REFERENCES


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