CHEBYSHEV APPROXIMATION WITH A NULL SPACE
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Abstract. Chebyshev approximation involving continuous functions vanishing on a closed set \( V \) is considered. The approximating families studied have the betweenness property. Examples are given of such families. A necessary and sufficient condition for uniqueness of best approximations is obtained.

Let \( X \) be a compact space and \( V \) be a closed subset of \( X \). Let \( C(V, X) \) be the space of continuous functions on \( X \) which vanish on \( V \). For \( g \in C(V, X) \) define
\[
\|g\| = \sup\{|g(x)|: x \in X\}.
\]
Let \( \mathcal{G} \) be a subset of \( C(V, X) \). The Chebyshev problem is: Given \( f \in C(V, X) \), find \( G^* \) in \( \mathcal{G} \) to minimize \( e(G) = \|f-G\| \). Such an element \( G^* \) is called a best approximation in \( \mathcal{G} \) to \( f \) on \( X \).

At least two cases of such a problem arise naturally, namely approximation with functions vanishing at zero and approximation with functions decaying to zero at infinity.

A seemingly more general problem is to approximate with functions which agree with \( v \in C(X) \) on a closed subset \( V \) of \( X \). This problem, however, reduces to the previous problem if we subtract \( v \) from all functions.

We consider the best approximation problem and in particular the uniqueness problem if \( \mathcal{G} \) has the betweenness property [1].

Definition. A family \( \mathcal{G} \) of continuous functions is said to have the betweenness property if for any two elements \( G_0 \) and \( G_1 \), there exists a \( \lambda \)-set \( \{H_\lambda\} \) of elements of \( \mathcal{G} \) such that \( H_0 = G_0 \), \( H_1 = G_1 \) and for all \( x \in X \), \( H_\lambda(x) \) is either a strictly monotonic continuous function of \( \lambda \) or a constant, \( 0 \leq \lambda \leq 1 \).

Example. Let \( \mathcal{G} \) be a linear subspace of \( C(V, X) \), then \( \mathcal{G} \) has the betweenness property, for a \( \lambda \)-set is given by \( H_\lambda = \lambda G_1 + (1-\lambda)G_0 \).

Example. Let \( P \) be a linear subspace of \( C(V, X) \) and \( Q \) a linear subspace of \( C(X) \) then \( \mathcal{G} = \{pq : p \in P, q \in Q, q > 0 \} \) is in \( C(V, X) \) and has the betweenness property [1, 152].
Lemma. Let $\sigma$ be a continuous strictly monotonic mapping of the real line into the real line such that $\sigma(0)=0$.

Let $\mathcal{G} \subset C(V, X)$ have the betweenness property. Then

$$\phi(\mathcal{G}) = \{\sigma(G) : G \in \mathcal{G}\} \subset C(V, X)$$

and has the betweenness property.

Proof. Let $\{H_\lambda\}$ be a $\lambda$-set for $G_0$ and $G_1$. Then $\{\sigma(H_\lambda)\}$ is a $\lambda$-set for $\sigma(G_0)$ and $\sigma(G_1)$.

Lemma. Let $\mathcal{G} \subset C(W, X)$ have the betweenness property and $s \in C(V, X)$. Then the set $s\mathcal{G}$ (consisting of products of $s$ and elements of $\mathcal{G}$) is in $C(W \cup V, X)$ and has the betweenness property.

The previous theory obtained for betweenness [1] gives a characterization of best approximations and an error-determining set on which best approximations agree. We must, however, develop a new theory for uniqueness.

Definition. $\mathcal{G} \subset C(V, X)$ has zero-sign compatibility with null space $V$ if for any two distinct elements $G$ and $H$, any closed subset $Z$ of the zeros of $G-H$ which contains no points of $V$ and for any $s \in C(V, X)$ taking values $-1$ or $+1$ on $Z$, there exists $F \in \mathcal{G}$ such that

$$\text{sgn}(F(x) - G(x)) = s(x), \quad x \in Z.$$  

Theorem. Let $\mathcal{G} \subset C(V, X)$ have the betweenness property. A necessary and sufficient condition that for every $f \in C(V, X)$ a best approximation is unique is that $\mathcal{G}$ have zero-sign compatibility with null space $V$.

The proof is the same as the proof of the corresponding result in [1].

The case where $\mathcal{G}$ is a finite-dimensional linear family is of particular interest. It can be shown using the above theorem that a necessary and sufficient condition for uniqueness is that $\mathcal{G}$ be a Haar subspace on $X \sim V$. Independent proofs of necessity and sufficiency are given in [3], [2], respectively.

References


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