

THE EQUATION OF RAMANUJAN-NAGELL AND $[y^2]$

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ABSTRACT. By arithmetizing Levi's constructive test for membership in $[y^2]$ we have translated the questions of whether a given power product is in $[y^2]$ to determining whether a certain product of matrices is the zero matrix. This leads to number-theoretic problems, including the diophantine equations of the title $2^n - 7 = x^2$.

Introduction. In the proof of the sufficiency of the low power theorem [4] and [10], one needs information concerning the differential ideal $[y^p]$, and Ritt suggests in his "Questions for Further Investigations" [10, p. 177] a further examination of this ideal. Levi [4] obtained a constructive test for determining whether any polynomial is in $[y^p]$, and we have arithmetized his method. Restricting ourselves to $[y^2]$ for simplicity, we show that the question of determining whether a power product belongs to $[y^2]$ can be translated into determining whether a certain product of matrices is the zero matrix which in turn can be translated into a number theoretic problem. In fact we encounter a problem stated by Ramanujan in 1913 [9], first solved by Nagell in 1948 [7], and solved several times since then [1], [2], [11], [12]. It may be of some interest to note that this problem, which arose in the study of error correcting codes [11], has now appeared in an investigation in differential algebra.

Notation. Let F be a field of characteristic zero, y a differential indeterminate over F , and $R = F\{y\}$, the differential ring of polynomials in y and its derivatives, with coefficients in F . Denoting differentiation by subscripts, if $P = y_{i_1} y_{i_2} \cdots y_{i_d}$, we say that P is of degree d and weight $w = \sum_{j=1}^d i_j$. Levi showed [4] that if $w < d(d-1)$ then $P \in [y^2]$, the smallest differential ideal in R containing y^2 , and for each $w \geq d(d-1)$ he gave examples of P which are not in the ideal. With the above power product, assuming $i_1 \leq i_2 \leq \cdots \leq i_d$, we associate the sequence (a_1, \cdots, a_d) where $a_k = \sum_{j=1}^k i_j - k(k-1)$, called the weight sequence.

Levi's condition can be stated as follows: the product P is in $[y^2]$ if some entry of its weight sequence is negative. The fact that this condition is not necessary was shown in [5], which also characterized all products which are in the ideal if their weight sequences contain no number larger than 2. An indication of some of the difficulties of a similar result for

Presented to the society, November 18, 1972; received by the editors January 22, 1973.
AMS (MOS) subject classifications (1970). Primary 12H05; Secondary 10A35, 10B15.

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power products, the elements of whose weight sequences are no larger than 3, is given in [8]. We present a new technique which can be applied to any weight sequence, but shall limit our discussion to those whose entries are ≤ 3 .

Sequences and the reduction process. We will show how Levi's reduction process for $[y^2]$ can be stated in terms of sequences. (It is easy to generalize this to $[y^p]$.) As described above, to every ordered monomial corresponds a sequence (a_1, \dots, a_n) . Conversely, to every weight sequence, (a_1, \dots, a_n) corresponds to the ordered monomial $y_{i_1}y_{i_2} \cdots y_{i_n}$ where $i_j = a_j - a_{j-1} + 2(j-1)$ if we allow the i_j to be negative and define $a_0 = 0$. If $2 + a_{i+1} + a_{i-1} - 2a_i \geq 0$ for $i = 1, 2, \dots, n-1$, then $i_1 \leq i_2 \leq \dots \leq i_n$. If for some k , $2 + a_{k+1} + a_{k-1} - 2a_k = t < 0$, it is easy to see that the sequence $(a_1, \dots, a_{k-1}, a_k + t, a_{k+1}, \dots, a_n)$ corresponds to

$$y_{i_1}y_{i_2} \cdots y_{i_{k-1}}y_{i_{k+1}}y_{i_k}y_{i_{k+2}} \cdots y_{i_n}.$$

By iterating this process, any sequence (a_1, \dots, a_n) can be put in canonical form, that is, so that in the corresponding ordered product, $i_1 \leq i_2 \leq \dots \leq i_n$.

We present a brief description of Levi's reduction process for $[y^2]$, and the simplification introduced in [5]. The product $Y = y_{i_1}y_{i_2} \cdots y_{i_n}$ is called an α -term if $i_1 + 2 \leq i_2 + 2 \leq \dots \leq i_{n-1} + 2 \leq i_n$, and the α -terms are linearly independent over F , modulo $[y^2]$. If $W = Y \cdot y_i y_{i+1}$, then by solving for $y_i y_{i+1}$ in $(y^2)_{2i+1}$ we obtain

$$W \equiv Y \sum_{j=0}^{i-1} \frac{2C_j^{2i+1}}{2C_i^{2i+1}} (-1)^j y_j y_{2i+1-j} \pmod{[y^2]}.$$

Similarly,

$$Y y_i^2 \equiv Y \sum_{j=0}^{i-1} \frac{C_j^{2i}}{C_i^{2i}} (-2)^j y_j y_{2i-j}.$$

In [5] it is shown we can suppress the numbers $2C_j^{2i+1}/2C_i^{2i+1}$, and C_j^{2i}/C_i^{2i} (which are there called first multipliers) and we write the above,

$$W \equiv^M Y \sum_{j=0}^{i-1} (-1)^j y_j y_{2i+1-j}, \quad \text{and} \quad Y y_i^2 \equiv^M Y \sum_{j=0}^{i-1} (-2)^j y_j y_{2i-j},$$

respectively. In [4] it is shown that after a finite number of steps any monomial is congruent, modulo $[y^2]$, to a linear combination of α -terms, and an element of R is in $[y^2]$ if and only if, in its expression as a linear combination of α -terms, all coefficients are zero. Since all of the congruences in this paper will be "multiplier" congruences, we drop the M and write \equiv , rather than \equiv^M .

Turning to sequences, we note that (a_1, \dots, a_n) , in canonical form, is an α -term if and only if $f(j) = a_{j+1} + a_{j-1} - 2a_j \geq 0$ for $j = 1, 2, \dots, n-1$. Assume $f(k) = -1$. Then, corresponding to the above, we have

$$(a_1, \dots, a_n) \equiv - \sum_{j=1}^{a_k} (a_1, \dots, a_{k-1}, a_k - j, a_{k+1}, \dots, a_n),$$

where, in general, the sequences on the right side of the congruence will not be in canonical form. It is easy to see that if the canonical form of $(a_1, \dots, a_{k-1}, a_k - r, a_{k+1}, \dots, a_n)$ has a negative entry, and hence is in the ideal, the same is true for all $j > r$, and the sum can be terminated with any such r . If $f(k) = -2$ (which corresponds to $i_k = i_{k+1}$), then we have

$$(a_1, \dots, a_n) \equiv -2 \sum_{j=1}^{a_k} (a_1, \dots, a_{k-1}, a_k - j, a_{k+1}, \dots, a_n).$$

This completes the description of the reduction process, for if (a_1, \dots, a_n) is in canonical form, then $f(k) \geq -2$ for $k = 1, 2, \dots, n-1$.

We finally note some results from [5, pp. 428-430], which will prove useful. If (A) and (A_i) are sequences, and $(A) \equiv \sum \alpha_i (A_i)$ for some rational numbers α_i , then $(0, A) \equiv \sum \alpha_i (0, A_i)$. Also, $(1, 1, A) \equiv -(0, 1, A)$; for $\varepsilon = 0, 1$, if $(A, \varepsilon) \equiv \alpha(0, \dots, 0, \varepsilon)$ then $(A, \varepsilon, B) \equiv \alpha(0, \dots, 0, \varepsilon, B)$; and if $(a_1, \dots, a_n, \varepsilon) \equiv \alpha(0, \dots, 0, \varepsilon)$ and $(\varepsilon, a_n, \dots, a_1, 0) = \beta(0, \dots, 0)$ then $\beta = 0$ if and only if $\alpha = 0$. It is clear that no confusion will arise if we delete a sequence of 0's at the beginning of a sequence; thus we write $(1, 2, 2, 2) \equiv -2(1, 2) + (1, 2, 2)$ rather than the more precise

$$-2(0, 0, 1, 2) + (0, 1, 2, 2).$$

The following relation will be useful:

$$\begin{aligned} (0, 2, 2, 2) &\equiv -2(1, 2, 2) - 2(0, 2, 2) \\ &\equiv 2(1, 1, 2) + 2(1, 0, 2) + 4(1, 2) + 4(0, 2) \\ &\equiv -2(1, 2) - 4(0, 2) + 4(1, 2) + 4(0, 2) \end{aligned}$$

or

$$(*) \quad (0, 2, 2, 2) \equiv 2(1, 2).$$

Matrices. As will be seen shortly, the sequence defined by $g(0) = 0$, $g(1) = 1$, $g(n+2) = g(n+1) - 2g(n)$ will be important for our work. It is easy to prove that $g(n) = 0$ if and only if $n = 0$, and we note that $|g(n)| = 1$

if $n=1, 2, 3, 5, 13$. We first describe the procedure for sequences (a_1, \dots, a_n) with $a_i \leq 2$.

The following relations, $(1, 2, 2) \equiv 0(1, 2) + (1, 2, 2)$, and $(1, 2, 2, 2) \equiv -2(1, 2) + (1, 2, 2)$ can be summarized by the matrix congruence

$$\begin{pmatrix} (1, 2, 2) \\ (1, 2, 2, 2) \end{pmatrix} \equiv M_2 \begin{pmatrix} (1, 2) \\ (1, 2, 2) \end{pmatrix} \quad \text{where } M_2 = \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix}.$$

Therefore, with $2_i=2$,

$$\begin{pmatrix} (1, 2_1, \dots, 2_{n+1}) \\ (1, 2_1, \dots, 2_{n+2}) \end{pmatrix} \equiv M_2^n \begin{pmatrix} (1, 2) \\ (1, 2, 2) \end{pmatrix}$$

where

$$M_2^n = \begin{pmatrix} (-1)^{n-1}2g(n-1) & (-1)^{n-1}g(n) \\ (-1)^n2g(n) & (-1)^ng(n+1) \end{pmatrix}.$$

Similarly,

$$\begin{pmatrix} (1, 2, 1) \\ (1, 2, 2, 1) \end{pmatrix} \equiv T_{21}((0, 1))$$

where $T_{21} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. Thus,

$$A = (1, 2_1, \dots, 2_{n+1}, 1) \equiv (1, 0)M_2^n T_{21}(0, 1) = 4(-1)^{n-1}g(n-1)(0, 1)$$

and since $(0, 1)$ is an α -term, A is in $[y^2]$ if and only if $g(n-1)=0$, i.e., $n=1$. Thus $(1, 2, 2, 1)$ is in $[y^2]$, and using $(*)$, we find $(0, 2, 2, 2, 2, 1)$ is also in $[y^2]$. Using a remark at the end of the previous section and $(*)$, we conclude $(1, 2, 2, 2, 2, 0)$ and $(2, 2, 2, 2, 2, 2, 0)$ are also in the ideal. In this way we easily obtain the main results of §4 in [5].

We turn now to weight sequences (a_1, \dots, a_n) where $a_i \leq 3$. As before, it is easy to show that

$$\begin{pmatrix} (1, 2, 3, 3) \\ (1, 2, 2, 3, 3) \\ (1, 3, 3) \end{pmatrix} \equiv M_3 \begin{pmatrix} (1, 2, 3) \\ (1, 2, 2, 3) \\ (1, 3) \end{pmatrix} \quad \text{where } M_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & -2 & -2 \end{pmatrix},$$

and that

$$M_3^n = \begin{pmatrix} (-1)^n & & 0 \\ 0 & -g(n+3) & 4g(n) \\ 0 & -2g(n) & 8g(n-3) \end{pmatrix}.$$

Similarly, we obtain the "transition" matrices,

$$\begin{pmatrix} (1, 2, 3, 2) \\ (1, 2, 2, 3, 2) \\ (1, 3, 2) \end{pmatrix} \equiv T_{32} \begin{pmatrix} (1, 2) \\ (1, 2, 2) \end{pmatrix} \quad \text{where } T_{32} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} (1, 2, 3, 1) \\ (1, 2, 2, 3, 1) \\ (1, 3, 1) \end{pmatrix} \equiv T_{31}((0, 1)) \quad \text{where } T_{31} = \begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} (1, 2, 0) \\ (1, 2, 2, 0) \end{pmatrix} \equiv T_{20}((0, 0)) \quad \text{where } T_{20} = 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

and

$$\begin{pmatrix} (1, 2, 3) \\ (1, 2, 2, 3) \\ (1, 3) \end{pmatrix} \equiv T_{23} \begin{pmatrix} (1, 2, 3) \\ (1, 2, 2, 3) \\ (1, 3) \end{pmatrix} \quad \text{where } T_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The following illustrates how the problem of membership in the ideal for sequences (a_1, \dots, a_n) with $a_i \leq 3$ can be stated in terms of the above matrices. Note $P = (1, 3_1, \dots, 3_{r+1}, 2)$ with $3_i = 3$ is in $[y^2]$ if and only if $(0, 0, 1)M_3^r T_{32} = (0, 4(g(r) + 2g(r-3)))$ is the zero matrix. Since $g(r) + 2g(r-3) = -g(r-2)$, it follows that $(1, 3_1, 3_2, \dots, 3_{r+1}, 2)$ is in the ideal if and only if $r=2$. Also, using $T_{20} = 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, we see that $(1, 3_1, \dots, 3_{n+1}, 2, 0) \in [y^2]$ only if $(1, 3_1, \dots, 3_{r+1}, 2) \in [y^2]$. Similarly, $(1, 0, 0)M_3^n T_{31}$ is the zero matrix for every n , and $(1, 0, 0)M_3^n$ is never the zero matrix (since M_3 is nonsingular). That is, $(1, 2, 3_1, \dots, 3_{n+1}, 1)$ is in the ideal for every $n \geq 0$, and $(1, 2, 3_1, \dots, 3_k)$ is never in the ideal. The conclusions in this paragraph contain the main results in [8].

We can now characterize all sequences (a_1, \dots, a_n) in the ideal where $1 < a_i \leq 3$ for $1 < i < n$, which do not start with $(2, 3, \dots)$ or $(2, 2, 3, \dots)$ or end with $(\dots, 3, 2, 0)$ or $(\dots, 3, 2, 2, 0)$.

THEOREM 1. *With $2_i = 2, 3_i = 3, n_i$ and $m_j \geq 0$, and $I = [y^2]$,*

- (1) $(1, 3_1, \dots, 3_{n+1}, 2) \in I$ if and only if $n=2$.
- (1, $2_1, \dots, 2_{n_1+1}, 3_1, \dots, 3_{m_1+1}, \dots, 3_1, \dots, 3_{m_k+1}, 1) \in I$ if and only if either $m_k = 2$ or $n_i = 0$ for every i .
- (2) $(1, 3_1, \dots, 3_{n_1+1}, 2_1, \dots, 2_{m_1+1}, 3_1, \dots, 3_{n_2+1}, \dots, 3_1, \dots, 3_{n_k+1}, 1) \in I$ if and only if one of n_1 and n_k is 2.
- (3) $(1, 2_1, \dots, 2_{n_1+1}, 3_1, \dots, 3_{m_1+1}, 2_1, \dots, 2_{n_2+1}, \dots, 3_1, \dots, 3_{m_k+1}, 2_1, \dots, 2_{n_{k+1}+1}, 1) \in I$ if and only if $\sum_{i=1}^{k+1} n_i = 1$.
- (4) $(1, 3_1, \dots, 3_{n+1}, 1) \in I$ if and only if $n=5$.

(5) $(2_1, \dots, 2_{n_1+3}, 3_1, \dots, 3_{m_1+1}, \dots, 3_1, \dots, 3_{m_k+1}, 2_1, \dots, 2_{n_k+1}, 1) \in I$ if and only if $\sum_{i=1}^{k+1} n_i = 1$.

(6) $(2_1, \dots, 2_{n_1+3}, 3_1, \dots, 3_{m_1+1}, \dots, 3_1, \dots, 3_{m_k+1}, 2_1, \dots, 2_{n_k+3}, 0) \in I$ if and only if $\sum_{i=1}^{k+1} n_i = 1$.

(7) In any of the above, replace (a_1, \dots, a_n) by $(a_n, a_{n-1}, \dots, a_1)$.

PROOF. Having just seen a proof of the first part of (1), we establish (3) before completing the proof of (1). With $n_1, n_2, \dots, m_1, m_2, \dots$ two sequences of nonnegative integers, let

$$(\alpha_k, \beta_k) = (1, 0) \left(\prod_{i=1}^{k-1} M_2^{n_i} T_{23} M_3^{m_i} T_{32} \right) M_2^{n_k}.$$

Then we find the following recursion relations for $k \geq 2$:

$$\begin{aligned} 2^{-1}\alpha_k &= (-1)^{m_{k-1}+n_{k-1}} 2\alpha_{k-1}g(n_k - 1) \\ &\quad + (-1)^{n_k} 2\beta_{k-1}g(n_k)(g(m_{k-1} + 3) + 2g(m_{k-1})), \\ 2^{-1}\beta_k &= (-1)^{m_{k-1}+n_{k-1}} \alpha_{k-1}g(n_k) \\ &\quad + (-1)^{n_k} \beta_{k-1}g(n_k + 1)(g(m_{k-1} + 3) + 2g(m_{k-1})). \end{aligned}$$

The determinant of this system of equations, with α_{k-1} and β_{k-1} the unknown, is

$$(-1)^{m_{k-1}-1}(g(m_{k-1} + 3) + 2g(m_{k-1})) \det \begin{pmatrix} 2g(n_k - 1) & 2g(n_k) \\ g(n_k) & g(n_k + 1) \end{pmatrix}$$

and this is not zero since $g(m_{k-1} + 3) + 2g(m_k) = -g(m_{k-1} + 1)$ and

$$\begin{pmatrix} 2g(n_k - 1) & 2g(n_k) \\ g(n_k) & g(n_k + 1) \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix}^{n_k}.$$

We see α_1 and β_1 cannot both be zero, hence, by induction the same is true for α_k and β_k . If $(\alpha_k, \beta_k)T_{21} = (0)$ then $\alpha_k = 0$. But, if $\alpha_k = 0$ then either $n_k = 1$ and $\beta_{k-1} = 0$ or $n_k = 0$ and $\alpha_{k-1} = 0$. Also, if $\beta_k = 0$ then $\beta_{k-1} = 0$ and $n_k = 0$. The conclusion (3) now follows readily. Since $(\alpha_k, \beta_k)T_{23}M_3^rT_{31} = 4\beta_k(g(r+3) + g(r)) = 4\beta_k(g(r-2))$, we see $(1, 2_1, \dots, 2_{n_1+1}, 3_1, \dots, 3_{m_1+1}, \dots, 3_1, \dots, 3_{m_k+1}, 1) \in I$ if and only if either $\beta_k = 0$ or $r = 2$. From this, one can easily complete the proof of (1).

To obtain the result in (2), with m_1, m_2, \dots and n_1, n_2, \dots sequences of nonnegative integers, we let $(\alpha_k, \beta_k, 0) = (0, 0, 1) \prod_{i=1}^k (M_3^{m_i} T_{32} M_2^{n_i} T_{23})$. Then

$$\alpha_1 = (-1)^{n_1} 8g(m_1 - 2)g(n_1), \quad \beta_1 = (-1)^{n_1} 4g(m_1 - 2)g(n_1 + 1),$$

and for $k \geq 1$,

$$\begin{aligned} \alpha_{k+1} &= 4(-1)^{n_k}((-1)^{m_k-1}g(n_k - 1)\alpha_k + g(n_k)g(m_k + 1)\beta_k), \\ \beta_{k+1} &= 2(-1)^{n_k}((-1)^{m_k-1}g(n_k)\alpha_k + g(n_k + 1)g(m_k + 1)\beta_k). \end{aligned}$$

By induction one can prove that if $\beta_1 \neq 0$, then for all $k > 1$, $\alpha_k/\beta_k = 2a_k/b_k$ where a_k and b_k are odd; i.e., if $\beta_1 \neq 0$, then $\beta_i \neq 0$ for all i . (It is clear that if $\beta_1 = 0$ then $\alpha_i = \beta_i = 0$ for all i .) Now,

$$\begin{aligned} (\alpha_k, \beta_k, 0)M_3^i T_{31} &= ((-1)^i \alpha_k, \beta_k(-g(t + 3)), \beta_4 g(t)) \begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix} \\ &= 4\beta_k(g(t + 3) + g(t)) = 4\beta_k(4g(t - 2)) \end{aligned}$$

and this is the zero matrix only if $\beta_k = 0$ or $t = 2$. This completes the proof of (2). (From (1), (2) states that a sequence of this type is in the ideal only if it has a factor in the ideal.)

To obtain (4), we note $(0, 0, 1)M_3^n T_{31} = (8(g(n) + g(n - 3))) = (8g(n - 5))$ which is zero only if $n = 5$. From equation (*) and the remarks at the end of the previous section, the results (5), (6), and (7) follow.

The equation of Ramanujan-Nagell. Ramanujan [9] conjectured that the diophantine equation $x^2 + 7 = 2^{n+2}$ had only 5 solutions corresponding to $n = 1, 2, 3, 5, 13$. This conjecture was first proved correct by Nagell [7]. An equivalent problem, which Mersenne numbers are triangular numbers, i.e., solve $2^m - 1 = k(k + 1)/2$, was solved by Browkin and Schinzel [1]. Another equivalent problem, for what value of n is $g(n) = \pm 1$ if $g(1) = g(2) = 1$ and $g(n + 2) = g(n + 1) - 2g(n)$, was solved by Chowla, Dunton and Lewis [2], and by Skolem, Chowla and Lewis [12]. In the proof of the following theorem, we encounter the same problem.

THEOREM 2. *With $2_i = 2$, and $3_i = 3, (2, 3_1, \dots, 3_{n+1}, 2_1, \dots, 2_{r+1}, 1) \in I$ if and only if:*

- $r = 2, n = 0$, or
- $r = 3, n = 1, 2, 4, 12$, or
- $r = 5, n = 3, 7$, or
- $r = 13, n = 11$.

PROOF. If $(2, 3_1, \dots, 3_{n+1}, 2_1, \dots, 2_{r+1}, 1) \in I$ then since $2(0, 2) = (1, 2, 2) - (1, 2)$ we should have $(-1, 1)T_{23}M_3^n T_{32}M_2^r T_{21} = (0)$. But this product equals $(-1)^{r+18}((-1)^{n-1}g(r-1) + g(r)g(n+1)) = 0$, and since $(g(r), g(r-1)) = 1$, we must have $g(r) = \pm 1$. In each of the references [2], [3], [6], [7], [11], [12], it is shown that this only occurs if $r = 1, 2, 3, 5, 13$. ($g(3) = g(5) = g(13) = -1$.)

If $r=1$, then $g(n+1)=0$ for which there is no nonnegative solution. If $r=2$, then $g(n+1)=(-1)^n$ and $n=1$. If $r=3$, we see that $-g(n+1)=(-1)^n$ which implies $n=1, 2, 4, 12$. If $r=5$, then $(-1)^{n-1}(-3)=g(n+1)$. To show that $n+1=4, 8$ are the only solutions we will show that $|g(k)|=3$ only for $k=4, 8$. Similarly, for $r=13$, we need $(-1)^{n-1}45=g(n+1)$ and we show that $|g(k)|=45$ only for $k=12$.

The proof of the theorem will be complete once we have proved the following lemma.

LEMMA. *If $g(1)=g(2)=1$ and $g(n+2)=g(n+1)-2g(n)$ then*

(a) *$|g(n)|=3$ if and only if $n=4, 8$.*

(b) *$|g(n)|=45$ if and only if $n=12$.*

PROOF. From the conditions on $g(n)$, it is well known that if m and n are positive integers and $m|n$ then $g(m)|g(n)$. Also, it is easy to show that $g(n+8)=g(n+4)-16g(n)$ for all n .

The remainders of $g(n)$ modulo 64 are 1, 1, -1, -3, -1, followed by a periodic pattern of 16 terms: 5, 7, -3, -17, -11, -23, -19, -1, -27, -25, 29, 15, 21, -9, 13, 31. Therefore if $g(n)=-3$, then $n=4$, or $n \equiv 8 \pmod{16}$; i.e., $n=4(4t+2)$. (We also note $g(n)$ can never be +3.) For $t > 0$, $g(4t+2) \neq \pm 1$ and since $4t+2|n$, $g(4t+2)|g(n)$ or $g(4t+2)=-3$. But $4t+2 \not\equiv 8 \pmod{16}$ and this contradiction completes the proof of (a).

Before turning to (b) we show that $|g(n)|=5$ only if $n=6$. From the remainders modulo 64, we see that if $|g(n)|=5$ then $g(n)=5$ and $n \equiv 6 \pmod{16}$; i.e., $n=2(8t+3)$. If $t > 0$, then $g(8t+3) \neq \pm 1$, and hence $g(8t+3)=5$. But $8t+3 \not\equiv 6 \pmod{16}$ and $|g(n)|=5$ only if $n=6$.

The proof of (b) can be done in a similar manner. We first show that $|g(n)|$ never takes on the value 9 or 15. If $|g(n)|=9$ then $g(n)=-9$ and $n \equiv 3 \pmod{16}$. The remainders of $g(4t+3)$ modulo 10 repeat in blocks of 6 and we find $n \equiv 24t+3=3(8t+1)$. For $t > 0$, $g(8t+1) \neq \pm 1, \pm 3$; hence $g(8t+1)=-9$ which implies $8t+1 \equiv 3 \pmod{16}$. This is a contradiction and we conclude $|g(n)|$ is never 9. If $|g(n)|=15$ then $g(n)=15$ and $n \equiv 17 \pmod{16}$. The remainders of $g(4k+1)$ modulo 17 repeat in blocks of 36 (most easily seen as 4 groups of 9 each) and if $g(n)=15$ then $n \equiv 21$ or $141 \pmod{144}$. This contradicts the above and we see $|g(n)|$ never takes on the value 15.

From the remainders modulo 64 we find $|g(n)|=45$ only if $g(n)=45$ and $n=16t+12=4(4t+3)$. For $t > 1$, $|g(4t+3)| \neq 1, 3, 5, 9, 15$. Hence $g(4t+3)=45$ which is impossible since $4t+3$ is odd; therefore we have shown $|g(n)|=45$ only if $n=12$.

Although other results similar to those in Theorem 2 can easily be obtained, we have not been successful in characterizing all sequences

$(a_1, \dots, a_n) \in I$ with $a_i \leq 3$, no less all sequences in I . Indeed, the appearance of the equation of Ramanujan-Nagell suggests that the search for a necessary and sufficient test for membership in $[y^2]$, stated in terms of the sequences (a_1, \dots, a_n) , may involve difficult, and possibly deep, number theoretic problems.

However, it may be of some interest to note that the same problem (the equation of Ramanujan-Nagell), which has attracted a fair amount of theoretical attention over the years, also arose in the study of error correcting codes, and has now reappeared in a problem in differential algebra. One wonders whether there is perhaps something fundamental about Ramanujan's problem, as well as when and where it may arise again.

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