A REMARK ON THE COMMUTATIVITY OF CERTAIN RINGS

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Abstract. In a recent paper [1] Gupta proved that a division ring satisfying the polynomial identity $xy^2x = yx^2y$ is commutative. In this note our goal is to prove the following: If $R$ is a semiprime ring with $xy^2x - yx^2y$ central in $R$, for all $x, y$ in $R$, then $R$ is commutative.

Throughout this paper a ring will mean an associative ring. A result of Gupta [1] asserts that a division ring satisfying the polynomial identity $xy^2x = yx^2y$ is commutative. Our present object is to generalize Gupta’s result as follows: If $R$ is a semiprime ring in which $xy^2x - yx^2y$ is in $Z$ (center of $R$) for every $x$ and $y$ in $R$, then $R$ is commutative. The author’s proof does not depend on any well-known theorem, however, and so provides a different and elementary proof of the above result.

We begin with the following:

Lemma. Let $R$ be a prime ring with $xy^2x - yx^2y$ in $Z$ for every $x$ and $y$ in $R$. Then $R$ is commutative.

Proof. First we shall prove that $Z \neq (0)$. So we assume that $Z = (0)$. Then we have

$$xy^2x = yx^2y,$$

for every $x$ and $y$ in $R$.

Replacing $y$ by $y + y^2$ we obtain

$$2xy^2x = y^2x^2y + yx^2y^2.$$  \(2\)

Since $y^2x^2y = y \cdot yx^2y = y \cdot xy^2x$, we get

$$2xy^2x = yx(y^2x + xy^2).$$  \(3\)

If the characteristic is 2, then (3) becomes $y(x(y^2x + xy^2)) = 0$. With $x = x + y$ this gives

$$y^2(x^2 + xy^2) = 0.$$  \(4\)

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Then with $x = rx$ we get

$$y^2(y^2rx + rxy^2) = 0.$$  

Since $y^2 \cdot y^2r = y^2 \cdot r y^2$ from (4), (5) becomes $y^2[r(y^2x + xy^2)] = 0$. We write this as $y^2R(y^2x + xy^2) = 0$. Since $R$ is prime, either $y^2 = 0$ or $y^2x + xy^2 = 0$, i.e. $y^2 \in Z = (0)$. Thus in either case $y^2 = 0$ for every $y$ in $R$.

If the characteristic is not 2, we replace $y$ by $y + y^3$ in (1) and get $2xy^4x = y^3x^2y + yx^2y^3$ or $2y^2x^2y^2 = y^2xy^2x + xy^2y^2$ (applying (1) to each term).

We write this as:

$$y^2x^2y^2 - y^2xy^2x = xy^2xy^2 - y^2x^2y^2$$

or

$$y^2x(xy^2 - y^2x) = (xy^2 - y^2x)xy^2.$$  

We replace $x$ by $x + y$ to get:

$$y^3(xy^2 - y^2x) = (xy^2 - y^2x)y^3,$$  

for every $x$ and $y$ in $R$.

Let $I_y$ be the inner derivation by $y^2$, i.e. $x \mapsto xy^2 - y^2x$, and $I_y$ be the inner derivation by $y^3$. Then (6) becomes $I_yI_y(x) = 0$. Thus the product of these derivations is again a derivation (the trivial one). Then by Theorem 1 in [3], we can conclude that either $y^2$ or $y^3$ is in $Z$, i.e. $= 0$. If it is $y^2 = 0$, then (2) becomes: $y^2x^2y + yx^2y^2 = 0$. We set $x = x + y$ to get $2y^2xy^2 = 0$ or $y^2xy^2 = 0$ or $y^2y^2 = 0$. Then $y = 0$.

Thus if $Z = 0$ then $y^2 = 0$ for every $y$ in $R$. Then $0 = (x + y)^3xy^2x$, or $xRx = 0$. Then $x = 0$, $R = 0$, a contradiction. Therefore $Z \neq (0)$.

Take $\lambda \neq 0$ in $Z$ and let $x = x + \lambda$ in $xy^2 - yx^2y$ in $Z$. We get $\lambda(xy^2 - 2xyy + y^2x)$ in $Z$. Since $R$ is prime, we must then have

$$xy^2 - 2yxy + y^2x \in Z,$$  

for, if $\lambda a$ is in $Z$, then $\lambda ab - b\lambda a = 0 = \lambda(ab - ba)$. Then $R \cdot (ab - ba) = 0 = \lambda R(ab - ba)$, and since $\neq 0$, we have $ab - ba = 0$, i.e. $a$ is in $Z$.

In (7) we let $x = xy$ and get $(xy^2 - 2yxy + y^2x)y$ in $Z$. Then $y$ is in $Z$ unless $y^2x^2 - 2yxy + y^2x = 0$. So if $y$ is not in $Z$, $xy^2 - 2yxy + y^2x = 0$ for every $x$ in $Z$, and if $y$ is in $Z$ then $xy^2 - 2yxy + y^2x$ is still 0. Therefore

$$xy^2 + y^2x = 2yxy.$$  

for every $x$ and $y$ in $R$.

If the characteristic is $\neq 2$, then by the sublemma [2, p. 5] $R$ is commutative.

If the characteristic is 2, then (8) becomes $xy^2 + y^2x = 0$ or $y^2$ is in $Z$ for every $y$ in $R$. Then $(x + y)^2 = x^2 + y^2 + xy + yx$ is in $Z$ or $xy + yx$ is in $Z$. Let $x = xy$ and get $(xy + yx)y$ is in $Z$. Then $y$ is in $Z$ unless $xy + yx = 0$, which also means $y$ is in $Z$. Thus $Z = R$ and $R$ is commutative.
Let $R$ be a semiprime ring in which $xy^2x - yx^2y$ is in $Z$ for every $x, y$ in $R$. Since $R$ is semiprime it is isomorphic to a subdirect sum of prime rings $R_a$ each of which, as a homomorphic image of $R$, satisfies the hypothesis placed on $R$. By the above Lemma the $R_a$ are commutative, hence $R$ is commutative. Thus, we have proved

**Theorem.** Let $R$ be a semiprime ring in which $xy^2x - yx^2y$ is in $Z$, the center of $R$, for every $x, y$ in $R$. Then $R$ is commutative.

Indeed, rings of $3 \times 3$ strictly upper triangular matrices over any ring satisfy the condition of the above theorem but these rings may not be commutative.

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**References**


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