A REMARK ON THE COMMUTATIVITY OF CERTAIN RINGS

RAM AWTAR

Abstract. In a recent paper [1] Gupta proved that a division ring satisfying the polynomial identity \( xy^2x = yx^2y \) is commutative. In this note our goal is to prove the following: If \( R \) is a semiprime ring with \( xy^2x - yx^2y \) central in \( R \), for all \( x, y \) in \( R \), then \( R \) is commutative.

Throughout this paper a ring will mean an associative ring. A result of Gupta [1] asserts that a division ring satisfying the polynomial identity \( xy^2x = yx^2y \) is commutative. Our present object is to generalize Gupta’s result as follows: If \( R \) is a semiprime ring in which \( xy^2x - yx^2y \) is in \( Z \) (center of \( R \)) for every \( x \) and \( y \) in \( R \), then \( R \) is commutative. The author’s proof does not depend on any well-known theorem, however, and so provides a different and elementary proof of the above result.

We begin with the following:

Lemma. Let \( R \) be a prime ring with \( xy^2x - yx^2y \) in \( Z \) for every \( x \) and \( y \) in \( R \). Then \( R \) is commutative.

Proof. First we shall prove that \( Z \neq 0 \). So we assume that \( Z = 0 \). Then we have

\[
xy^2x = yx^2y, \quad \text{for every } x \text{ and } y \text{ in } R.
\]

Replacing \( y \) by \( y + y^2 \) we obtain

\[
2xy^3x = yx(y^2x + xy^2).
\]

Since \( y^2x = y \cdot yx^2y = y \cdot xy^2x \), we get

\[
2xy^3x = xy(y^2x + xy^2).
\]

If the characteristic is 2, then (3) becomes \( yx(y^2x + xy^2) = 0 \). With \( x = x + y \) this gives

\[
y^2(y^2x + xy^2) = 0.
\]
Then with \( x = rx \) we get

\[(5) \quad y^2(r^2x + rxy^2) = 0.\]

Since \( y^2 \cdot y^2r = y^2 \cdot ry^2 \) from (4), (5) becomes \( y^2[r(y^2x + xy^2)] = 0. \) We write this as \( y^2R(y^2x + xy^2) = 0. \) Since \( R \) is prime, either \( y^2 = 0 \) or \( y^2x + xy^2 = 0, \) i.e. \( y^2 \in Z = \{0\}. \) Thus in either case \( y^2 = 0 \) for every \( y \) in \( R. \)

If the characteristic is not 2, we replace \( y \) by \( y + y^3 \) in (1) and get \( 2y^4x = y^2x^2y + yx^2y^3 \) or \( 2y^2x^2y^2 = y^2xy^2x + xy^2x^2 \) (applying (1) to each term).

We write this as:

\[y^2x^2y^2 - y^2xy^2x = xy^2xy^2 - y^2x^2y^2\]

or

\[y^2x(y^2 - y^2x) = (xy^2 - y^2)xy^2.\]

We replace \( x \) by \( x + y \) to get:

\[(6) \quad y^3(xy^2 - y^2x) = (xy^2 - y^2)xy^3, \text{ for every } x \text{ and } y \text{ in } R.\]

Let \( I\) be the inner derivation by \( y^2, \) i.e. \( x \rightarrow xy^2 - y^2x, \) and \( I\) be the inner derivation by \( y^3. \) Then (6) becomes \( I\) for every \( x \) in \( R. \) Thus the product of these derivations is again a derivation (the trivial one). Then by Theorem 1 in [3], we can conclude that either \( y^2 \) or \( y^3 \) is in \( Z, \) i.e. \( = 0. \) If it is \( y^3 = 0, \) then (2) becomes:

\[y^2x^2y + yx^2y^3 = 0. \]

We set \( x = x + y \) to get \( 2y^2xy^2 = 0 \) or \( y^2xy^2 = 0 \) or \( y^2xy^2 = 0. \) Then \( y^2 = 0. \)

Thus if \( Z = 0 \) then \( y^2 = 0 \) for every \( y \) in \( R. \) Then \( 0 = (x + y)^3x = xyx, \) or \( xRx = 0. \) Then \( x = 0, \) \( R = 0, \) a contradiction. Therefore \( Z \neq \{0\}. \)

Take \( \lambda \neq 0 \) in \( Z \) and let \( x + \lambda \) in \( xy^2x - yx^2y \) in \( Z. \) We get \( \lambda(x^2 - 2yxy + y^2x) \) in \( Z. \) Since \( R \) is prime, we must then have

\[(7) \quad xy^2 - 2yxy + y^2x \quad \text{in } Z,\]

for, if \( \lambda \) is in \( Z, \) then \( \lambda ab - b\lambda a = 0 = \lambda(ab - ba). \) Then \( R \cdot \lambda(ab - ba) = 0 = \lambda R(ab - ba), \) and since \( \lambda \neq 0, \) we have \( ab - ba = 0, \) i.e. \( a \) is in \( Z. \)

In (7) we let \( x = xy \) and get \( (xy^2 - 2yxy + y^2x) \) in \( Z. \) Then \( y \) is in \( Z \) unless \( xy^2 - 2yxy + y^2x = 0. \) So if \( y \) is not in \( Z, \) \( xy^2 - 2yxy + y^2x = 0 \) for every \( x \) in \( R, \) and if \( y \) is in \( Z \) then \( xy^2 - 2yxy + y^2x \) is still 0. Therefore

\[(8) \quad xy^2 + y^2x = 2yxy \quad \text{for every } x \text{ and } y \text{ in } R.\]

If the characteristic is \( \neq 2, \) then by the sublemma [2, p. 5] \( R \) is commutative.

If the characteristic is 2, then (8) becomes \( xy^2 + y^2x = 0 \) or \( y^2 \text{ is in } Z \)

for every \( y \) in \( R. \) Then \( (x + y)^2 = x^2 + y^2x + yx + xy^2 \)

is in \( Z \) or \( xy + yx \) is in \( Z. \) Let \( x = xy \) and get \( (xy + yx)y = xy + yx \) is in \( Z. \) Then \( y \) is in \( Z \) unless \( xy + yx = 0, \)

which also means \( y \) is in \( Z. \) Thus \( Z = R \) and \( R \) is commutative.
Let $R$ be a semiprime ring in which $xy^2x - yx^2y$ is in $Z$ for every $x, y$ in $R$. Since $R$ is semiprime it is isomorphic to a subdirect sum of prime rings $R_\alpha$ each of which, as a homomorphic image of $R$, satisfies the hypothesis placed on $R$. By the above Lemma the $R_\alpha$ are commutative, hence $R$ is commutative. Thus, we have proved

**Theorem.** Let $R$ be a semiprime ring in which $xy^2x - yx^2y$ is in $Z$, the center of $R$, for every $x, y$ in $R$. Then $R$ is commutative.

Indeed, rings of $3 \times 3$ strictly upper triangular matrices over any ring satisfy the condition of the above theorem but these rings may not be commutative.

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**References**


Department of Mathematics, Aligarh Muslim University, Aligarh (U.P.), India