FINITELY GENERATED STEADY \( \mathfrak{N} \)-SEMIGROUPS

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Abstract. In this paper the author proves that \( S \) is a finitely generated steady \( \mathfrak{N} \)-semigroup if and only if \( S \) is isomorphic to the direct product of a finite abelian group and the infinite cyclic semigroup; and also studies the homomorphisms of a finitely generated steady \( \mathfrak{N} \)-semigroup into another.

1. Introduction. By an \( \mathfrak{N} \)-semigroup we mean a commutative archimedean cancellative semigroup without idempotent. Following Petrich [9] an \( \mathfrak{N} \)-semigroup \( S \) is called steady if \( S \) cannot be embedded into another \( \mathfrak{N} \)-semigroup as a proper ideal. In this note the author determines the structure of finitely generated steady \( \mathfrak{N} \)-semigroups. Such a semigroup is isomorphic to the direct product of a finite abelian group and the infinite cyclic semigroup.

2. Preliminaries. Let \( P \) denote the set of all positive integers and \( P^0 \) the set of all nonnegative integers. The structure of \( \mathfrak{N} \)-semigroups was given by the author:

Theorem 1 ([3], [11]). Let \( G \) be an abelian group and \( I: G \times G \to P^0 \) be a function satisfying

\begin{align*}
(1.1) & \quad I(\alpha, \beta) = I(\beta, \alpha) \text{ for all } \alpha, \beta \in G. \\
(1.2) & \quad I(\alpha, \beta) + I(\alpha \beta, \gamma) = I(\alpha, \beta \gamma) + I(\beta, \gamma) \text{ for all } \alpha, \beta, \gamma \in G. \\
(1.3) & \quad I(e, \alpha) = 1 \text{ (} e \text{ being the identity of } G \text{) for all } \alpha \in G. \\
(1.4) & \quad \text{For each } \alpha \in G \text{ there is } m \in P \text{ such that } I(\alpha, a^m) > 0.
\end{align*}

Let \( S \) be the set of all ordered pairs \( \{x, \alpha\}, x \in P^0, \alpha \in G \). Define an operation in \( S \) by

\[ \{x, \alpha\} \{y, \beta\} = \{x + y + I(\alpha, \beta), \alpha \beta\}. \]

Then \( S \) is an \( \mathfrak{N} \)-semigroup, denoted by \( S = (G; I) \). Every \( \mathfrak{N} \)-semigroup can be obtained in this manner.

Let \( D \) be an \( \mathfrak{N} \)-semigroup and let \( a \in D \). Define a relation \( \rho \) on \( D \) by \( x \rho y \) if and only if \( a^m x = a^n y \) for some \( m, n \in P \). Then \( \rho \) is a congruence and \( G = D/\rho \) is an abelian group and there exists \( I: G \times G \to P^0 \) such that
$D \cong (G; I)$. $G$ is called the structure group of $D$ with respect to $a$. Thus $G$ and $I$ depend on an element $a$; so we denote these by $G_a$ and $I_a$ respectively if it is necessary to specify $a$. An $\mathcal{R}$-semigroup $S$ is called power joined if for every $a, b \in S$ there are $m, n \in P$ such that $a^m = b^n$.

**Proposition 2.** (2.1) ([1], [2]). An $\mathcal{R}$-semigroup $S=(G; I)$ is power joined if and only if $G$ is periodic.

(2.2) ([1], [2], [6]). $S=(G; I)$ is finitely generated if and only if $G$ is finite.

Let $S=(G; I)$ be a power joined $\mathcal{R}$-semigroup. Let $R$ denote the set of positive rational numbers. Define a function $\varphi : G \to R$ by

$$\varphi(a) = \frac{1}{n} \sum_{i=1}^{n} I(a, x^i)$$

where $n$ is the order of an element $a$ of $G$.

**Proposition 3** [10]. The function $\varphi$ satisfies the following conditions:

(3.1) $\varphi(e) = 1$, $e$ the identity of $G$.

(3.2) $\varphi(\alpha) + \varphi(\beta) - \varphi(\alpha \beta)$ is a nonnegative integer, and

(3.3) $I(\alpha, \beta) = \varphi(\alpha) + \varphi(\beta) - \varphi(\alpha \beta)$.

If $S$ is finitely generated, equivalently, $G$ is finite, then

$$\varphi(\alpha) = \frac{1}{|G|} \sum_{\xi \in G} I(\alpha, \xi) \quad \text{where } |G| \text{ is the order of } G.$$

There is a one-to-one correspondence between $I$ and $\varphi$ for a fixed $G$ if $G$ is periodic. Thus $S$ is determined by $G$ and $\varphi$, and it is denoted by $S=(G; \varphi)$. The notation $"S=(G; I)=(G; \varphi)"$ means that $\varphi$ corresponds to $I$. Let $a$ be an element of $S$. The function $\varphi$ corresponding to $I_a$ is denoted by $\varphi_a$.

If $G$ is finite, we can choose an element $a$ of $S$ such that

$$|G_a| \leq \sum_{\xi \in G_a} I_a(\alpha, \xi) \quad \text{for all } \alpha \in G_a.$$

(See [7].) Then $(G_a; I_a)$ or $(G_a; \varphi_a)$ is called a canonical representation of $S$. Speaking of $\varphi$, $(G; \varphi)$ is canonical if and only if

$$\varphi(\alpha) \geq 1 \quad \text{for all } \alpha \in G.$$

Let $S=(G; I)$ and $\Lambda(S)$ be the semigroup of all translations of $S$. Then $\Lambda(S)$ is a commutative cancellative semigroup. Let $\Gamma(S)$ be the subsemigroup of $\Lambda(S)$ consisting of all inner translations of $S$, and $\Psi(S)$ the archimedean component of $\Lambda(S)$ containing $\Gamma(S)$. Note that $S \cong \Gamma(S)$, that $\Psi(S)$ is also an $\mathcal{R}$-semigroup, and that $\Psi(S) = \{ \lambda \in \Lambda(S) : \lambda^n \in \Gamma(S) \}$ for some $n \in P$. Each element of $\Lambda(S)$ is determined by a pair $(m, \alpha) \in P^n \times G$ where if $m=0$, we further require that $I(\alpha, \xi) > 0$ for all $\xi \in G$.  

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The translation corresponding to \((m, \alpha)\) is denoted by \(\lambda_{(m, \alpha)}\), and it takes an element \(\{x, \xi\}\) of \(S\) to the element \(\{x+m+I(\alpha, \xi)-1, \alpha \xi\}\) in \(S\). The multiplication in \(\Lambda(S)\) is given by
\[
\lambda_{(m, \alpha)} \cdot \lambda_{(n, \beta)} = \lambda_{(m+n+I(\alpha, \beta)-1, \alpha \beta)}.
\]
Then we can see that \(\Gamma(S) = \{\lambda_{(m, \alpha)} : m > 0, \alpha \in G\}\) and that \(\Psi(S) = \Gamma(S) \cup A\) where \(A = \{\lambda_{(m, \alpha)} : I(\alpha, \xi) > 0\) for all \(\xi \in G\) and \(I(\alpha, \alpha^m) > 1\) for some \(m \in P\}\).

See [4], [5] with respect to the translations of \(\mathfrak{N}\)-semigroups.

**Theorem 4 (Petrich [9]).** The following conditions on an \(\mathfrak{N}\)-semigroup \(S\) are equivalent:

1. For any \(a, b \in S\), \(aS \subseteq bS\) and \(a^nS \subseteq b^nS\) imply \(a \in bS\).
2. \(\Psi(S) = \Gamma(S)\).
3. \(S\) cannot be embedded into an \(\mathfrak{N}\)-semigroup as a proper ideal.

If an \(\mathfrak{N}\)-semigroup \(S\) satisfies one of (4.1), (4.2) and (4.3), \(S\) is called steady.

The condition (4.2) is equivalent to \(A = \emptyset\). Hence we get the following lemma.

**Lemma 5.** An \(\mathfrak{N}\)-semigroup \(S = (G; I)\) is steady if and only if \(I(\alpha, \xi) > 0\) for all \(\xi \in G\) implies \(I(\alpha, \alpha^m) = 1\) for all \(m \in P\).

**3. Main result.** In this paper we treat only finitely generated \(\mathfrak{N}\)-semigroups.

**Theorem 6.** Let \(S = (G; I)\) be a finitely generated \(\mathfrak{N}\)-semigroup. The following are equivalent:

1. \(I(\alpha, \xi) > 0\) for all \(\xi \in G\) implies \(I(\alpha, \alpha^m) = 1\) for all \(m \in P\).
2. \(I(\alpha, \xi) > 0\) for all \(\xi \in G\) implies \(\phi(\alpha) = 1\).
3. \(\phi(\alpha) \leq 1\) for all \(\alpha \in G\).
4. \(I(\alpha, \xi) > 0\) for all \(\xi \in G\) implies \(I(\alpha, \xi) = 1\) for all \(\xi \in G\).

**Proof.** (6.1)\(\Rightarrow\) (6.2): Obvious by (2.3).

(6.2)\(\Rightarrow\) (6.3): Let \(\phi(\beta)\) be the maximum of \(\{\phi(\xi) : \xi \in G\}\). Suppose \(I(\beta, \xi) = \phi(\beta) + \phi(\xi) - \phi(\beta \xi) = 0\) for some \(\xi \in G\). Then \(\phi(\beta) < \phi(\beta \xi)\). This is in contradiction to maximality of \(\phi(\beta)\). Accordingly \(I(\beta, \xi) > 0\) for all \(\xi \in G\). By (6.2), \(\phi(\beta) = 1\), hence \(\phi(\alpha) \leq 1\) for all \(\alpha \in G\).

(6.3)\(\Rightarrow\) (6.4): Assume \(I(\alpha, \xi) > 0\) for all \(\xi \in G\). By (6.3) we have \(0 < I(\alpha, \xi) = \phi(\alpha) + \phi(\xi) - \phi(\alpha \xi) \leq 2 - \phi(\alpha \xi) < 2\). Therefore \(I(\alpha, \xi) = 1\).

(6.4)\(\Rightarrow\) (6.1): Obvious.

As far as finitely generated \(\mathfrak{N}\)-semigroups are concerned, each of (6.1) through (6.4) is a necessary and sufficient condition for \(S = (G; I)\) to be steady because of Lemma 5. Although the functions \(I\) and \(\phi\) depend on the choice of standard elements, the condition in Lemma 5, (6.1), (6.2), (6.3)
Theorem 7. A finitely generated \( \mathfrak{N} \)-semigroup \( S \) is steady if and only if \( S \) is isomorphic to the direct product of the positive integer semigroup \( P \) and a finite abelian group.

Proof. Assume that \( S \) is steady. Let \( S=(G; I)\) be a canonical representation of \( S \). By (3.6) and (6.3), \( I(a) = 1 \) for all \( a \in G \). Then \( I(a, b) = \varphi(a) + \varphi(b) - \varphi(a\cdot b) = 1 \) for all \( a, b \in G \). With respect to the representation of \( S=(G; I) \),

\[
\{x, a\} \cdot \{y, b\} = \{x+y+1, ab\}.
\]

Define a map \( f:S \to P \times G \) by \( \{x, a\} \mapsto (x+1, a) \). We can easily see that \( f \) is an isomorphism.

Conversely let \( S=P \times G \) where \( G \) is a finite abelian group. Let \( a=(1, e) \), \( e \) being the identity of \( G \). Then all elements prime to \( a \) have the form \( (1, a), a \in G \).

\[
(1, a)(1, b) = (1, e)(1, x\cdot b).
\]

From this it follows that the structure group \( G_a \) is isomorphic to \( G \) and \( I_a(a, b) = 1 \) for all \( a, b \in G_a \). By Theorem 6 and Lemma 5 we conclude that \( S \) is steady.

Remark. McAlister and O'Carroll characterized \( P \times G \) in the more general case, that is, they proved in [8] that a finitely generated cancellative semigroup \( S \) without idempotent is isomorphic to \( P \times G \) if and only if \( S^2=Sa \) for all \( a \in S \setminus S^2 \).

Remark. The equivalence of (6.1), (6.2) and (6.4) is valid even if \( S=(G; I) \) is power joined.

4. Homomorphisms. Let \( S=P \times G \) where \( G \) is a finite abelian group. It is easy to see that the structure group \( G_{(m,a)} \) of \( S \) with respect to an element \( (m, a) \) of \( S \) has order \( m \cdot |G| \) and \( G_{(1,a)} \) has the smallest order of the structure groups of \( S \). Furthermore \( G_{(1,a)} \cong G_{(1,1)} \cong G \). Let \( G \) and \( H \) be finite abelian groups. It follows that \( P \times G \cong P \times H \) if and only if \( G \cong H \).

More generally we consider homomorphisms of one steady \( \mathfrak{N} \)-semigroup into another. Let \( S=P \times G, T=P \times H \). Let \( (m, a) \) and \( [x, \xi] \) denote elements of \( S \) and \( T \) respectively. Assume that \( f \) is a homomorphism of \( S \) into \( T \). For each \( a \in G \), let \( f(1, a)=[p(a), q(a)] \) where \( p:G \to P, q:G \to H \). Then

\[
f(m, a) = f((1, e)^{m-1}(1, a)) = (f(1, e))^{m-1}f(1, a) = [(m-1)p(e) + p(a), q(e)^{m-1}q(a)]
\]
where \( e \) is the identity of \( G \). Likewise

\[
f(n, \beta) = [(n - 1)p(e) + p(\beta), q(e)^{n-1}q(\beta)],
\]
and

\[
f((m, \alpha)(n, \beta)) = [(m + n - 1)p(e) + p(\alpha\beta), q(e)^{m+n-1}q(\alpha\beta)].
\]

From \( f((m, \alpha)(n, \beta)) = f(m, \alpha)f(n, \beta) \), we get

\[
p(\alpha) + p(\beta) = p(\alpha\beta) + p(e), \quad q(\alpha)q(\beta) = q(\alpha\beta)q(e)
\]
for all \( \alpha, \beta \in G \).

Let \( r(\alpha) = p(\alpha) - p(e) \) and \( s(\alpha) = q(\alpha)q(e)^{-1} \). Then we have \( r(\alpha) + r(\beta) = r(\alpha\beta) \), \( s(\alpha)s(\beta) = s(\alpha\beta) \), that is, \( r \) is a homomorphism of \( G \) into \( Z \) where \( Z \) is the group of integers under addition and \( s \) is a homomorphism of \( G \) into \( H \). However, since \( G \) is finite, \( r(\alpha) = 0 \) for all \( \alpha \in G \), hence \( p(\alpha) = p(e) \) for all \( \alpha \in G \). Let \( t = p(e), \sigma = q(e) \). We get \( f(m, \alpha) = [tm, \sigma^m\sigma(\alpha)] \).

Conversely let \( s \) be a homomorphism of \( G \) into \( H \), \( \sigma \) a fixed element of \( H \) and \( t \) a fixed positive integer. Define \( f: S \rightarrow T \) by

\[
f(m, \alpha) = [tm, \sigma^m\sigma(\alpha)].
\]

Then it is easy to show that \( f \) is a homomorphism. Thus all homomorphisms of \( S \) into \( T \) are determined by \( t \in P, \sigma \in H \), and \( s \in \text{Hom}(G, H) \). Moreover we see from (8) that \( f \) is one-to-one if and only if \( s \) is one-to-one; \( f \) is onto if and only if \( t = 1 \) and \( s \) is onto. Let \( \text{Hom}(S, T) \) denote the semigroup of all homomorphisms of \( S \) into \( T \) in the usual sense. Clearly \( \text{Hom}(S, T) \neq \emptyset \). Consequently we have the following theorem.

**Theorem 9.** (9.1) \( \text{Hom}(P \times G, P \times H) \cong P \times H \times \text{Hom}(G, H) \) hence it is a finitely generated steady \( \mathcal{R} \)-semigroup.

(9.2) \( P \times H \) is a homomorphic image of \( P \times G \) if and only if \( G \) is homomorphic onto \( H \).

(9.3) \( P \times G \) is isomorphic into (onto) \( P \times H \) if and only if \( G \) is isomorphic into (onto) \( H \).

Let \( |G| < \infty \) and \( S = (G; I) \). According to [6], \( S \) is isomorphic to a subdirect product of a positive integer additive semigroup and \( G \), hence \( S \) can be embedded into \( P \times G \) in the natural way. Consider the category of the embeddings of \( S \) into finitely generated \( \mathcal{R} \)-semigroups. Even if \( S = (G; I) \) is canonical, the embedding \( S \rightarrow P \times G \) need not be a universal object. (See the example below.) We state the following theorem without detailed proof.

**Theorem 10.** Let \( S = (G; I) = (G; \varphi), \ |G| < \infty, \) and let \( G_0 = \{ \alpha \in G: \varphi(\alpha) \in P \} \). Then \( G_0 \) is a subgroup of \( G \) and \( S \) is isomorphic to a subsemigroup of \( P \times G_0 \) and the embedding \( S \rightarrow P \times G_0 \) is a universal repelling object in the category of the embeddings of \( S \) into finitely generated steady \( \mathcal{R} \)-semigroups.
$Z \times G_0$ is isomorphic to the quotient group of $S$, and $G_0$ is isomorphic to the torsion subgroup of $Z \times G_0$. It is easy to see that $S \rightarrow P \times G_0$ is a universal object, but $G_0$ need not be a structure group of $S$. (See the example below.) Related to Theorem 10, see [8] and [12].

**Example.** Let $G$ be the Klein four group: $a^2 = \beta^2 = e$. Define $\varphi$ by

$$
\varphi(e) = \varphi(\alpha) = 1, \quad \varphi(\beta) = \frac{1}{2}, \quad \varphi(\alpha \beta) = \frac{1}{2}.
$$

Let $S = (G; \varphi)$. $S$ has two canonical representations with respect to $\{0, e\}$ and $\{0, \alpha\}$. Then $G_{\{0,e\}} \cong G$ but $G_{\{0,\alpha\}}$ is a cyclic group of order 4, hence $P \times G \not\cong P \times G_{\{0,\alpha\}}$. Since $|G_{\{0,\beta\}| = 6$, $P \times G$ cannot be embedded into $P \times G_{\{0,\beta\}}$ but $S$ can be embedded into $P \times G_{\{0,\beta\}}$. $G_0$ consists of $e$ and $\alpha$, and $G_0$ is not a structure group of $S$.

**References**


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