A NOTE ON STRASSEN'S VERSION OF THE LAW OF THE ITERATED LOGARITHM

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Abstract. Strassen's law of the iterated logarithm is extended to stationary ergodic martingales and to a non-identically-distributed case.

1. Introduction. V. Strassen [6] proved that if \( \{x_n\} \) are i.i.d.r.v.'s with \( E x = 0, E x^2 = 1 \), then for a continuous map \( \phi \) from \( C[0, 1] \) to \( \mathbb{R}^1 \), the sequence \( \phi(\eta_n) \) is relatively compact and the set of its limit points coincides with \( \phi(K) \) where \( \eta_n \) is obtained by linearly interpolating \((2n \log \log n)^{-1/2}S_i\) at \( t = i/n \), \( i = 1, 2, \ldots, n \), and \( S_i = x_1 + \cdots + x_i \).

\( K \) is the set of absolutely continuous functions \( f \in C[0, 1] \) such that \( f(0) = 0 \) and \( \int_1^1 |f'(t)|^2 \, dt \leq 1 \).

The key idea of the paper is that Strassen's theorem holds for any random sequence for which there is a Skorohod embedding in Brownian motion with stopping times \( \tau_n \) satisfying the strong law of large numbers. Two examples are given: one is the stationary ergodic martingales and the other a non-identically-distributed case.

As an easy extension of the corollary of Strassen [6], we can obtain in particular if \( \{x_i, F_i, i \geq 1\} \) is a stationary ergodic stochastic sequence \( E(x_i|F_{i-1}) = 0 \) a.s. for all \( i \geq 1 \) and \( E x_i^2 = 1 \), then

\[
(1) \quad \limsup \frac{S_n}{(2n \log \log n)^{-1/2}} = 1 \text{ a.s.}
\]

and

\[
(2) \quad \limsup \frac{\frac{1}{n} \sum_{i=1}^{n} S_i}{(\frac{3}{2} n \log \log n)^{1/2}} = 1 \text{ a.s.}
\]
(1) is recently obtained by W. F. Stout [7] by a different method and
(2) is an extension of the iterated logarithm for “Cesàro sums” considered
by Gal and Stackelberg [5], and Gaposhkin [2].

2. Results.

Lemmal 1. If \( \{\tau_i\} \), the sequence of stopping times of the Skorohod
representation theorem, satisfies the strong law of large numbers namely
\[
(\tau_1 + \cdots + \tau_n)/n \to 1 \text{ a.s.,}
\]
then there is a probability space with a Brownian motion \( B(t) \) and a sequence
\( \{S^n\}_1^\infty \) having the same distribution as \( \{S^n\}_1^\infty \) such that
\[
(3) \quad \sup_{r \leq t} \frac{|S^n - B(r)|}{(2t \log \log t)^{1/2}} \to 0 \quad \text{a.s. as } t \to \infty.
\]

Proof. Follows from [6].

Theorem 1. Strassen’s theorem [6] holds for a stationary ergodic mar-
tingale difference sequence \( \{x_n, F_n, n \geq 1\} \) with \( E(x_1^2) = 1 \).

To facilitate the proof of Theorems 1 and 2 let us state some known
results from Jonas’s thesis [3] and often we shall sketch a proof.

A generalization of Skorohod’s theorem. Let \( (\mathcal{A}) \) \( \hat{X}_1, \hat{X}_2, \cdots \) be
a sequence of random variables on a probability space \( (\Omega, \mathcal{B}, P) \) for which
\( E(\hat{X}_n^2|\hat{X}_1, \cdots, \hat{X}_n) \) exists for all \( n \) and \( E(\hat{X}_n|\hat{X}_1, \cdots, \hat{X}_n) = 0 \) a.s.

Let \( (\Omega, \mathcal{B}, P) \) be a probability space with the following properties:
(1) There exists a Brownian motion \( B = B(t, \omega) \) on \( (\Omega, \mathcal{B}, P) \).
(2) There exist random variables independent of the Brownian motion
\( B, Y_1, Y_2, \cdots \) on \( (\Omega, \mathcal{B}, P) \) which are pairwise independent and distrib-
uted uniformly over the interval \([0, 1]\).

Then there exists a sequence of nonnegative random variables \( \tau_1, \tau_2, \cdots \)
on \( (\Omega, \mathcal{B}, P) \) and \( (B) \) random variables \( X_1, X_2, \cdots \) on \( (\Omega, \mathcal{B}, P) \), such
that r.v.’s of the sequence \( (\mathcal{A}) \) and \( (B) \) have the same distribution and
\( \sum^n X_i = B(\sum^n \tau_i) \) a.s.

Moreover, if \( \mathcal{A}_n = \mathcal{B}(X_1, X_2, \cdots, X_n, B(t), 0 \leq t \leq \sum^n \tau_i) \), i.e. the
\( \sigma \)-field generated by \( X_1, \cdots, X_n \) and \( B(t) (0 \leq t \leq \sum^n \tau_i) \), then the following
hold.
(1) \( \tau_n \) is \( \mathcal{A}_n \)-measurable.
(2) For each \( s > 0 \), \( B_n(s) = B(\sum^n \tau_i + s) - B(\sum^n \tau_i) \) is independent
of \( \mathcal{A}_n \).
(3) \( E(\tau_n|\mathcal{A}_{n-1}) \) exists and \( E(\tau_n|\mathcal{A}_{n-1}) = E(X_n^2|\mathcal{A}_{n-1}) \) a.s. = \( E(X_n^2|X_1, X_2, \cdots, X_{n-1}) \).
If \( k > 0 \) and if \( E(X_{n+1}^2 \mid X_1, \ldots, X_{n-1}) \) exists, then \( E(\tau_n^k \mid \mathcal{A}_{n-1}) \) exists too, and further

\[
E(\tau_n^k \mid \mathcal{A}_{n-1}) \leq L_k E(X_n^{2k} \mid \mathcal{A}_{n-1}) = L_k E(X_n^{2k} \mid X_1, \ldots, X_{n-1}) \quad \text{a.s.}
\]

where \( L_k \) is a constant depending only on \( k \).

**Sketch of proof.** Like Skorohod [4, pp. 163–164], let us state

**Lemma 2.** Suppose that \( B(t) \) is a Brownian motion process for \( t \geq 0 \) and \( B(0) = 0 \) and \( \tau_{n+1} \) is the smallest root of \((B_n(t)-a)(B_n(t)-b)=0\), where \( a < 0 < b \). Then for every \( \lambda > 0 \),

\[
E(\exp(-\lambda \tau_{n+1}) I_{[B_n(\tau_{n+1})=b]} \mid \mathcal{A}_n) = \frac{\sinh b(2\lambda)^{1/2}}{\sinh b - a(2\lambda)^{1/2}} \quad \text{a.s.}
\]

and

\[
E(\exp(-\lambda \tau_{n+1}) I_{[B_n(\tau_{n+1})=a]} \mid \mathcal{A}_n) = -\frac{\sinh a(2\lambda)^{1/2}}{\sinh(b - a)(2\lambda)^{1/2}} \quad \text{a.s.}
\]

We can prove the lemma and the theorem by induction i.e. we would assume that \( \tau_1, \tau_2, \ldots, \tau_n \) and \( X_1, X_2, \ldots, X_n \) are already constructed having the desired properties. We now consider the construction of \( \tau_{n+1} \).

Let \( f_a = \exp(-\lambda \tau_{n+1}) I_{[B_n(\tau_{n+1})=a]} \). By assumption \( B_n(t) \) is independent of \( \mathfrak{A}_n \) and, by definition, of \( \tau_{n+1} \). \( \mathfrak{B}(f_a) \subset \mathfrak{B}(B_n(t), t \geq 0) \); therefore \( \mathfrak{B}(f_a) \) is independent of \( \tau_n \). Like Skorohod [4, p. 166] let us state a

**Corollary.** If \( B_n(t), \tau_{n+1} \) is as above, then for \( \lambda > 0 \) the following hold:

1. \[
E(\exp(-\lambda \tau_{n+1}) \mid \mathcal{A}_n) = \frac{\sinh b(2\lambda)^{1/2} - \sinh a(2\lambda)^{1/2}}{\sinh(b - a)(2\lambda)^{1/2}} \quad \text{a.s.};
\]
2. \[
E(I_{[B_n(\tau_{n+1})=a]} \mid \mathcal{A}_n) = \frac{b}{b - a} \quad \text{and}
\]
3. \[
E(I_{[B_n(\tau_{n+1})=b]} \mid \mathcal{A}_n) = -\frac{a}{b - a};
\]
4. For each \( k > 0 \), there exists a constant \( C_k \) depending on \( k \) only such that

\[
E(\tau_n^k \mid \mathcal{A}_n) \leq C_k ab(b-a)^{2k-2}.
\]

Here also the \( \sigma \)-field generated by those r.v.'s whose conditional expectation will be constructed is contained in the \( \sigma \)-field \( \mathfrak{B}(B_n(t), t \geq 0) \) and hence independent of \( \mathcal{A}_n \).

Let \( F_n(x, u) \) be the conditional distribution function of \( \tilde{X}_{n+1} \) given \((X_1, X_2, \ldots, X_n)=x\). If the jumps (or discontinuities) of \( F \) are joined by vertical lines and the graph so obtained is reversed then we obtain a function \( f_n(x, t) \) on \( R^n \times [0, 1] \) to \( R \). For fixed \( x \) this \( f_n(x, t) \), except for at most countable points \( t \), is defined everywhere in \([0, 1]\). These points are precisely those \( t \)'s for which \( F_n(x, u)=t \). \( f_n(x, t) \) is the distribution
function mapping. Since \( E(\hat{X}_{n+1} | \hat{X}_1, \cdots, \hat{X}_n) = 0 \), like Skorohod, we get
\[
\int_{a_n(x)}^{b_n(x)} f_n(x, t) dt = -\int_{\hat{a}_n(x)}^{\hat{b}_n(x)} f_n(x, t) dt
\]
where
\[
\alpha_n(x) = F_n(x, 0) + \frac{1}{2} \left( \lim_{z \to 0^+} F_n(x, z) - \lim_{z \to 0^-} F_n(x, z) \right)
\]
and a function \( G_n : R^n \times [0, 1] \to [0, 1] \) defined by
\[
\int_{a_n(x)}^{b_n(x)} f_n(x, t) dt = -\int_{G_n(x, y)}^{G_n(x, z)} f_n(x, t) dt
\]
and a function \( G_n : R^n \times [0, 1] \to [0, 1] \) defined by
\[
\int_{a_n(x)}^{b_n(x)} f_n(x, t) dt = -\int_{G_n(x, y)}^{G_n(x, z)} f_n(x, t) dt
\]
and also \( G_n \) has the property
\[
G_n(x, G_n(x, y)) = x \quad \text{a.s.} \quad (y \in [0, 1]), \quad G_n(x, \alpha_n(x)) = \alpha_n(x) \quad \text{a.s.}
\]
Let \( Y_{n+1} \) be the \((n+1)\)th uniformly distributed random variable on \((\Omega, \mathcal{B}, P)\). Then we can define two mappings \( X_{n+1}^a \) and \( X_{n+1}^b \) from \( R^n \times [0, 1] \) to \( R \) by
\[
X_{n+1}^a(x, \omega) = f_n(x, Y_{n+1}(\omega)), \quad X_{n+1}^b(x, \omega) = f_n(x, G_n(x, Y_{n+1}(\omega))).
\]
and \( X_{n+1}^a \) and \( X_{n+1}^b \) are \( \mathcal{B} \times \mathcal{B}(Y_{n+1}) \) measurable random variables where \( \mathcal{B} \) is the usual Borel field on \( R^n \) and \( \mathcal{B}(Y_{n+1}) \) is the Borel field defined by \( Y_{n+1} \). Like Skorohod [4, p. 167] we state the lemma.

**Lemma 3.** Let \( B_n(t) \) be independent of \( \mathcal{A}_n \) and \( Y_{n+1} \). Let \( T(x, \eta) \), \( x \in R^n, \eta \in [0, 1] \) be the smallest solution of
\[
(B_n(t, \eta) - X_{n+1}^a(x, \eta))(B_n(t, \eta) - X_{n+1}^b(x, \eta)) = 0.
\]
Further let \( T_{n+1}(\omega) = T(X_1(\omega), \cdots, X_n(\omega), \omega) \) and \( X_{n+1} = B_n(T_{n+1}) \), then \( T_{n+1} \) is \( \mathcal{A}_n \) measurable and \( X_1, X_2, \cdots, X_{n+1} \) and \( \hat{X}_1, \hat{X}_2, \cdots, \hat{X}_{n+1} \) have the same distribution.

Let \( F_\eta \) be the \( \sigma \)-field generated by \( X_1, X_2, \cdots, X_\eta \) and \( F_0 = \mathcal{A}_0 = \{ \phi, \Omega \} \).  

**Proof of Theorem 1.** Now by the Birkhoff ergodic theorem
\[
n^{-1}[E(\tau_n | \mathcal{A}_{n-1}) + E(\tau_{n-1} | \mathcal{A}_{n-2}) + \cdots + E(\tau_2 | \mathcal{A}_1) + E(\tau_1 | \mathcal{A}_0)]
\]
\[
\quad \rightarrow \frac{1}{n} \sum_{i=1}^{n} E(X_i^2 | F_{i-1}) \rightarrow EX_1^2 = 1 \quad \text{a.s.};
\]
now it is enough to show that
\[
\frac{1}{n} \sum_{i=1}^{n} [\tau_i - E(\tau_i | \mathcal{A}_{i-1})] \rightarrow 0 \quad \text{a.s.};
\]
using Theorem 5 of Chow [1], the result will follow if

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} E(\tau_n - E(\tau_n \mid \mathcal{F}_{n-1}))^2 < \infty. \]

Now

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} E(\tau_n - E(\tau_n \mid \mathcal{F}_{n-1}))^2 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} E\tau_n^2 \leq L_2 \sum_{n=1}^{\infty} \frac{1}{n^2} E\mathcal{X}_n^4. \]

Since \((X_n, i \geq 1)\) is a stationary ergodic sequence with \(E(X_i \mid F_{i-1}) = 0\) a.s. for all \(i \geq 2\), proceeding as in Skorohod [4] we see that \((\tau_i, i \geq 1)\) are stationary. Since \((Y_n, n \geq 1)\) are independent identically distributed uniform random variables on \([0, 1]\) and \((X_n, n \geq 1)\) are stationary ergodic stochastic sequences, \(\{X_n^a\}\) and \(\{X_n^b\}\) are also stationary ergodic stochastic sequences.

Now by a truncated argument, e.g., considering \(X^*_n = X_n I_{\{|X_n| < \varepsilon \sqrt{n}\}}\), (6) is true. Therefore \(n^{-1}(\tau_1 + \tau_2 + \cdots + \tau_n) \rightarrow 1\) a.s. So by Lemma 1,

\[ \sup_{t \geq 1} \frac{|S_{lr} - B_{lr}|}{(2t \log \log t)^{1/2}} \rightarrow 0 \quad \text{a.s. as } t \rightarrow \infty. \]

So the theorem follows from the Corollary to Theorem 3 of [6].

**Theorem 2.** Strassen's theorem [6] holds if \(x_1, x_2, \cdots\) are independent r.v.'s with \(E x_i = 0, E x_i^2 = 1\) and

\[ \sum_{k=2}^{\infty} E \left| x_k \right|^{2+\delta} < \infty, \quad 2 \geq \delta > 0 \]

(in particular if \(E \left| x_k \right|^{2+\delta} \leq C\) for all \(k\)).

**Proof.** Now

\[ E \left| \tau_k^{-1} \right|^{1+\delta/2} \leq 2^\delta (E \tau_k^{1+\delta/2} + 1) \leq 2^\delta (C_\delta E \left| x_k \right|^{2+\delta} + 1). \]

So by Theorem 5 of Chow [1]

\[ \sum_{k=2}^{\infty} E \left| x_k \right|^{2+\delta} < \infty \quad \text{implies} \quad \frac{1}{n} \sum_{i=1}^{n} (\tau_i - E\tau_i) \rightarrow 0 \quad \text{a.s.} \]

Therefore \(\{\tau_i\}\) satisfies the conditions of Lemma 1.

**Remarks 1.** Gaposhkin [2] proved that if \(x_1, x_2, \cdots\) are independent r.v.'s with \(|x_k| \leq C\) a.s. and \(E x_k^2 = 1, E x_k = 0\) for all \(k = 1, 2, \cdots\) then

\[ \limsup \frac{1}{\log n} \sum_{k=1}^{n} \left(1 - \frac{k}{n}\right)^{x_k} \left(\frac{2}{n \log \log n}\right)^{-1/2} = 1 \quad \text{a.s., } \alpha > 0. \]

We shall extend this result under the conditions of Theorems 1 and 2.
Let $S_0=0$. By Abel's sum

$$
\sum_{k=1}^{n} \left(1 - \frac{k}{n}\right)^{x_k} = \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right)^{x_k} - \left(1 - \frac{k+1}{n}\right)^{x_k} S_k
$$

Taking $f(t)=a(1-t)^{a-1}$ on p. 218 of Strassen [6] we get

$$
\limsup_{n \to \infty} 2^{1/2} \left(2^{n^3 \log \log n} \right)^{1/2} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) S_k = \left(\frac{1}{2\alpha + 1}\right)^{1/2}.
$$

Similarly Theorem 2 and Theorem 3 of Gaposhkin [2] may be extended.

2. We conjecture that if $x_1, x_2, \cdots$ is a martingale difference sequence with $E_{x_1}=0$, $E_{x_2^n}=1$ for all $n$ and $\{x_{n_1}\}_{n=1}^{\infty}$ is uniformly integrable, then the law of the iterated logarithm holds.

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