A NOTE ON STRASSEN'S VERSION OF THE LAW OF THE ITERATED LOGARITHM

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Abstract. Strassen's law of the iterated logarithm is extended to stationary ergodic martingales and to a non-identically-distributed case.

1. Introduction. V. Strassen [6] proved that if \( \{x_n\} \) are i.i.d.r.v.'s with \( Ex=0, Ex^2=1 \), then for a continuous map \( \phi \) from \( C[0, 1] \) to \( \mathbb{R}^1 \), the sequence \( \phi(\eta_n) \) is relatively compact and the set of its limit points coincides with \( \phi(K) \) where \( \eta_n \) is obtained by linearly interpolating \( (2n \log \log n)^{-1/2} S_i \) at \( t = i/n, i = 1, 2, \ldots, n \), and \( S_i = x_1 + \cdots + x_i \).

\( K \) is the set of absolutely continuous functions \( f \in C[0, 1] \) such that \( f(0)=0 \) and \( \int_0^1 |f'(t)|^2 \, dt \leq 1 \).

The key idea of the paper is that Strassen's theorem holds for any random sequence for which there is a Skorohod embedding in Brownian motion with stopping times \( \tau_n \) satisfying the strong law of large numbers. Two examples are given: one is the stationary ergodic martingales and the other a non-identically-distributed case.

As an easy extension of the corollary of Strassen [6], we can obtain in particular if \( \{x_i, F_i, i \geq 1\} \) is a stationary ergodic stochastic sequence \( E(x_i|F_{i-1})=0 \) a.s. for all \( i \geq 1 \) and \( Ex_i^2=1 \), then

\[
\limsup_n S_n(2n \log \log n)^{-1/2} = 1 \text{ a.s.}
\]

and

\[
\limsup_n \frac{1}{n} \sum_{i=1}^n S_i \left( \frac{3}{8} n \log \log n \right)^{1/2} = 1 \text{ a.s.}
\]
(1) is recently obtained by W. F. Stout [7] by a different method and
(2) is an extension of the iterated logarithm for “Cesàro sums” considered
by Gal and Stackelberg [5], and Gaposhkin [2].

2. Results.

Lemmma 1. If \{\tau_i\}, the sequence of stopping times of the Skorohod
representation theorem, satisfies the strong law of large numbers namely

\[(\tau_1 + \cdots + \tau_n)/k \to 1 \text{ a.s.},\]

then there is a probability space with a Brownian motion \(B(t)\) and a sequence
\(\{S_n\}_1^\infty\) having the same distribution as \(\{S_n\}_1^\infty\) such that

\[
\sup_{t \leq t} \frac{|S_t - B(t)|}{(2t \log \log t)^{1/2}} \to 0 \text{ a.s. as } t \to \infty.
\]

Proof. Follows from [6].

Theorem 1. Strassen’s theorem [6] holds for a stationary ergodic mar-
tingale difference sequence \(\{x_n, F_n, n \geq 1\} \text{ with } Ex_1^2 = 1\).

To facilitate the proof of Theorems 1 and 2 let us state some known
results from Jonas’s thesis [3] and often we shall sketch a proof.

A generalization of Skorohod’s theorem. Let (A) \(\hat{X}_1, \hat{X}_2, \cdots\) be a
sequence of random variables on a probability space \((\Omega, \mathcal{F}, P)\) for which
\(E(\hat{X}_n^2|\hat{X}_1, \cdots, \hat{X}_n)\) exists for all \(n\) and \(E(\hat{X}_n|\hat{X}_1, \cdots, \hat{X}_n) = 0\) a.s.

Let \((\Omega, \mathcal{F}, P)\) be a probability space with the following properties:

1. There exists a Brownian motion \(B = B(t, \omega)\) on \((\Omega, \mathcal{F}, P)\).

2. There exist random variables independent of the Brownian motion
\(B, Y_1, Y_2, \cdots\) on \((\Omega, \mathcal{F}, P)\) which are pairwise independent and distrib-
uted uniformly over the interval \([0, 1]\).

Then there exists a sequence of nonnegative random variables \(\tau_1, \tau_2, \cdots\)
on \((\Omega, \mathcal{F}, P)\) and (B) random variables \(X_1, X_2, \cdots\) on \((\Omega, \mathcal{F}, P)\), such
that r.v.’s of the sequence (A) and (B) have the same distribution and
\(\sum_1^n X_t = B(\sum_1^n \tau_t)\) a.s.

Moreover, if \(A_n = \mathcal{B}(X_1, X_2, \cdots, X_n, B(t), 0 \leq t \leq \sum_1^n \tau_j)\), i.e. the
\(\sigma\)-field generated by \(X_1, \cdots, X_n\) and \(B(t)\) \((0 \leq t \leq \sum_1^n \tau_j)\), then the following
hold.

1. \(\tau_n\) is \(A_n\)-measurable.

2. For each \(s > 0\), \(B_n(s) = B(\sum_1^n \tau_j + s) - B(\sum_1^n \tau_j)\) is independent
of \(A_n\).

3. \(E(\tau_n|A_{n-1})\) exists and \(E(\tau_n|A_{n-1}) = E(X^2_n|A_{n-1})\) a.s. = \(E(X^2_n|X_1, X_2, \cdots, X_{n-1})\).
(4) If \(k > 0\) and if \(E(X_n^{2k} | X_1, \ldots, X_{n-1})\) exists, then \(E(\tau_n^k | \mathcal{A}_{n-1})\) exists too, and further

\[
E(\tau_n^k | \mathcal{A}_{n-1}) \leq L_k E(X_n^{2k} | \mathcal{A}_{n-1}) = L_k E(X_n^{2k} | X_1, \ldots, X_{n-1}) \quad \text{a.s.}
\]

where \(L_k\) is a constant depending only on \(k\).

**Sketch of proof.** Like Skorohod [4, pp. 163–164], let us state

**Lemma 2.** Suppose that \(B(t)\) is a Brownian motion process for \(t \geq 0\) and \(B(0) = 0\) and \(\tau_{n+1}\) is the smallest root of \((B_n(t) - a)(B_n(t) - b) = 0\), where \(a < 0 < b\). Then for every \(\lambda > 0\),

\[
E(\exp(-\lambda \tau_n^{n+1}) | I_{B_n(\tau_n+1) = b}) | \mathcal{A}_n) = \frac{b(2\lambda)^{1/2}}{\sinh b(2\lambda)^{1/2}} a.s.
\]

and

\[
E(\exp(-\lambda \tau_n^{n+1}) | I_{B_n(\tau_n+1) = a}) | \mathcal{A}_n) = \frac{-a(2\lambda)^{1/2}}{\sinh (b - a)(2\lambda)^{1/2}} a.s.
\]

We can prove the lemma and the theorem by induction i.e. we would assume that \(\tau_1, \tau_2, \ldots, \tau_n\) and \(X_1, X_2, \ldots, X_n\) are already constructed having the desired properties. We now consider the construction of \(\tau_{n+1}\).

Let \(f_a = \exp(-\lambda \tau_{n+1}) I_{B_n(\tau_{n+1}) = a}\). By assumption \(B_n(t)\) is independent of \(\mathcal{A}_n\) and, by definition, of \(\tau_{n+1}\). \(\mathcal{B}(f_a) \subset \mathcal{B}(B_n(t), t \geq 0)\); therefore \(\mathcal{B}(f_a)\) is independent of \(\tau_n\). Like Skorohod [4, p. 166] let us state a

**Corollary.** If \(B_n(t), \tau_{n+1}\) is as above, then for \(\lambda > 0\) the following hold:

1. \(E(\exp(-\lambda \tau_{n+1}) | \mathcal{A}_n) = \frac{\sinh b(2\lambda)^{1/2} - \sinh a(2\lambda)^{1/2}}{\sinh (b - a)(2\lambda)^{1/2}} a.s.;\)

2. \(E(I_{B_n(\tau_{n+1}) = a}) | \mathcal{A}_n) = b/(b - a)\) and

3. \(E(I_{B_n(\tau_{n+1}) = b}) | \mathcal{A}_n) = -a/(b - a);\)

4. For each \(k > 0\), there exists a constant \(C_k\) depending on \(k\) only such that \(E(\tau_n^k | \mathcal{A}_n) \leq C_k ab(b - a)^{2k-2}.\)

Here also the \(\sigma\)-field generated by those r.v.’s whose conditional expectation will be constructed is contained in the \(\sigma\)-field \(\mathcal{B}(B_n(t), t \geq 0)\) and hence independent of \(\mathcal{A}_n\).

Let \(F_n(x, u)\) be the conditional distribution function of \(\hat{X}_{n+1}\) given \((X_1, X_2, \ldots, X_n) = x\). If the jumps (or discontinuities) of \(F\) are joined by vertical lines and the graph so obtained is reversed then we obtain a function \(f_n(x, t)\) on \(R^n \times [0, 1]\) to \(R\). For fixed \(x\) this \(f_n(x, t)\), except for at most countable points \(t\), is defined everywhere in \([0, 1]\). These points are precisely those \(t\)’s for which \(F_n(x, u) = t\). \(f_n(x, t)\) is the distribution
function mapping. Since $E(\hat{X}_{n+1}|\hat{X}_1, \cdots, \hat{X}_n)=0$, like Skorohod, we get
\[ \int_{\alpha_n(x)}^u f_n(x, t) \, dt = -\int_{\alpha_n(x)}^u f_n(x, t) \, dt \]
where
\[ \alpha_n(x) = F_n(x, 0) + \frac{1}{2} \left( \lim_{z \to 0} F_n(x, z) - \lim_{z \to 0} F_n(x, z) \right) \]
and a function $G_n: R^n \times [0, 1] \to [0, 1]$ defined by
\[ \int_{\alpha_n(x)}^u f_n(x, t) \, dt = -\int_{\alpha_n(x)}^u f_n(x, t) \, dt \]
and also $G_n$ has the property
\[ G_n(x, G_n(x, y)) = x \quad \text{a.s.} \quad (y \in [0, 1]), \quad G_n(x, \alpha_n(x)) = \alpha_n(x) \quad \text{a.s.} \]
Let $Y_{n+1}$ be the $(n+1)$th uniformly distributed random variable on $(\Omega, \mathcal{B}, P)$. Then we can define two mappings $X_{n+1}^a$ and $X_{n+1}^b$ from $R^n \times [0, 1]$ to $R$ by
\[ X_{n+1}^a(x, \omega) = f_n(x, Y_{n+1}(\omega)), \quad X_{n+1}^b(x, \omega) = f_n(x, G_n(x, Y_{n+1}(\omega)). \]
$X_{n+1}^a$ and $X_{n+1}^b$ are $\mathcal{B} \times \mathcal{B}(Y_{n+1})$ measurable random variables where $\mathcal{B}$ is the usual Borel field on $R^n$ and $\mathcal{B}(Y_{n+1})$ is the Borel field defined by $Y_{n+1}$. Like Skorohod [4, p. 167] we state the lemma.

**Lemma 3.** Let $B_n(t)$ be independent of $\mathcal{A}$ and $Y_{n+1}$. Let $T(x, \eta)$, $x \in R^n$, $\eta \in [0, 1]$ be the smallest solution of
\[ (4) \quad (B_{n+1}(t, \eta) - X_{n+1}^a(x, \eta))(B_{n+1}(t, \eta) - X_{n+1}^b(x, \eta)) = 0. \]
Further let $T_{n+1}(\omega) = T(X_1(\omega), \cdots, X_n(\omega), \omega)$ and $X_{n+1} = B_n(T_{n+1})$, then $T_{n+1}$ is $\mathcal{A}$ measurable and $X_1, X_2, \cdots, X_{n+1}$ and $\hat{X}_1, \hat{X}_2, \cdots, \hat{X}_{n+1}$ have the same distribution.

Let $F_i$ be the $\sigma$-field generated by $X_1, X_2, \cdots, X_i$ and $F_0 = \mathcal{A}_0 = \{\phi, \Omega\}$. **Proof of Theorem 1.** Now by the Birkhoff ergodic theorem
\[ n^{-1}[E(\tau_n | \mathcal{A}_{n-1}) + E(\tau_{n-1} | \mathcal{A}_{n-2}) + \cdots + E(\tau_2 | \mathcal{A}_1) + E(\tau_1 | \mathcal{A}_0)] \]
\[ \to \frac{1}{n} \sum_{i=1}^n E(X_i^2 | F_{i-1}) \to EX_1^2 = 1 \quad \text{a.s.}; \]
now it is enough to show that
\[ \frac{1}{n} \sum_{i=1}^n [\tau_i - E(\tau_i | \mathcal{A}_{i-1})] \to 0 \quad \text{a.s.}; \]
using Theorem 5 of Chow [1], the result will follow if

\[
(6) \quad \sum_{n=1}^{\infty} \frac{1}{n^2} E(\tau_n - E(\tau_n | \mathcal{G}_{n-1}))^2 < \infty.
\]

Now

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} E(\tau_n - E(\tau_n | \mathcal{G}_{n-1}))^2 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} E\tau_n^2 \leq L_2 \sum_{n=1}^{\infty} \frac{1}{n^2} E X_n^4.
\]

Since \((X_i, i \geq 1)\) is a stationary ergodic sequence with \(E(X_i | F_{i-1}) = 0\) a.s. for all \(i \geq 2\), proceeding as in Skorohod [4] we see that \((\tau_i, i \geq 1)\) are stationary. Since \((Y_n, n \geq 1)\) are independent identically distributed uniform random variables on \([0, 1]\) and \((X_n, n \geq 1)\) are stationary ergodic stochastic sequences, \(\{X_n^a\}\) and \(\{X_n^b\}\) are also stationary ergodic stochastic sequences.

Now by a truncated argument, e.g., considering \(X_n^* = X_n I_{\{|X_n| < \varepsilon \sqrt{n}\}}\), (6) is true. Therefore \(n^{-1}(\tau_1 + \tau_2 + \cdots + \tau_n) \rightarrow 1\) a.s. So by Lemma 1,

\[
\sup_{t \in T} \frac{|S_t - B(t)|}{(2t \log \log t)^{1/2}} \rightarrow 0 \quad \text{a.s. as } t \rightarrow \infty.
\]

So the theorem follows from the Corollary to Theorem 3 of [6].

**Theorem 2.** Strassen's theorem [6] holds if \(x_1, x_2, \cdots\) are independent r.v.'s with \(E x_i = 0, E x_i^2 = 1\) and

\[
\sum_{k=2}^{\infty} E |x_k|^{2+\delta} < \infty, \quad 2 \geq \delta > 0
\]

(in particular if \(E |x_k|^{2+\delta} \leq C\) for all \(k\)).

**Proof.** Now

\[
E |\tau_k^{-1}|^{1+\delta/2} \leq 2^{\delta}(E \tau_k^{1+\delta/2} + 1) \leq 2^{\delta}(C \delta E |x_k|^{2+\delta} + 1).
\]

So by Theorem 5 of Chow [1]

\[
\sum_{k=2}^{\infty} E |x_k|^{2+\delta} < \infty \quad \text{implies} \quad \frac{1}{n} \sum_{i=1}^{n} (\tau_i - E\tau_i) \rightarrow 0 \quad \text{a.s.}
\]

Therefore \(\{\tau_i\}\) satisfies the conditions of Lemma 1.

**Remarks 1.** Gaposhkin [2] proved that if \(x_1, x_2, \cdots\) are independent r.v.'s with \(|x_k| \leq C\) a.s. and \(E x_k^2 = 1, E x_k = 0\) for all \(k = 1, 2, \cdots\) then

\[
\limsup \sum_{k=1}^{n} \left(1 - \frac{k}{n}\right)^{\alpha} x_k \left(\frac{2}{2\alpha + 1} n \log \log n\right)^{-1/2} = 1 \quad \text{a.s., } \alpha > 0.
\]

We shall extend this result under the conditions of Theorems 1 and 2.
Let \( S_0 = 0 \). By Abel’s sum
\[
\sum_{k=1}^{n} \left(1 - \frac{k}{n}\right)x_k = \sum_{k=0}^{n-1} \left(\left(1 - \frac{k}{n}\right) - \left(1 - \frac{k+1}{n}\right)\right) S_k \\
\cong \alpha \frac{1}{n} \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right)^{x-1} S_k.
\]
Taking \( f(t) = \alpha (1-t)^{x-1} \) on p. 218 of Strassen [6] we get
\[
\limsup_{n \to \infty} \frac{1}{2n^3 \log \log n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) S_k = \left(\frac{1}{2\alpha + 1}\right)^{1/2}.
\]
Similarly Theorem 2 and Theorem 3 of Gaposhkin [2] may be extended.

2. We conjecture that if \( x_1, x_2, \ldots \) is a martingale difference sequence with \( E(x_1) = 0, E(x_2^2) = 1 \) for all \( n \) and \( \{x_2^n\}_{n=1}^{\infty} \) is uniformly integrable, then the law of the iterated logarithm holds.

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