LIE ALGEBRA REPRESENTATIONS
OF DIMENSION \( p - 1 \)

HELMUT STRADE

Abstract. A semisimple Lie algebra over an algebraically closed field of characteristic \( p > 2 \) admitting a faithful representation of dimension \( p - 1 \) is either a direct sum of classical algebras or the Witt algebra.

I. Introduction. In this note we discuss simple Lie algebras of characteristic \( p > 2 \) by use of filtrations. By a filtration of length \( r \) of a simple Lie algebra \( L \) is meant a sequence of subalgebras

\[ L = L_{-1} \supset L_0 \supset \cdots \supset L_r \supset L_{r+1} = (0), \quad L_r \neq (0), \]

where \( L_0 \) is a given proper subalgebra and the remaining algebras are inductively defined by

\[ L_{i+1} := \{ x \in L_i, xL \subseteq L_i \}. \]

It follows that \( L_i L_j \subseteq L_{i+j} \) for \( i+j \geq -1 \).

A filtration is said to be long if \( r \geq 2 \), short if \( r = 0 \), maximal if \( L_0 \) is a maximal algebra and nilpotent if \( L_0 \) acts nilpotently on \( L \). Kostrikin calls Lie algebras which possess a long filtration strongly degenerate.

An important result about Lie algebras which possess a long filtration is mentioned in [7] by Kostrikin: There exists an element \( c \neq 0 \) such that

\[ R_c R_{x_1} \cdots R_{x_{p-1}} R_c = 0. \]

Here \( R_x \) means the right multiplication \( y R_x := yx \).

Proposition 1 expresses this in terms of the length of possible filtrations.

Using this result and a characterization of Lie algebras which possess no long filtration ([7], [3], [9]), we prove in this note the following

Theorem. Let \( L \) be a semisimple Lie algebra over an algebraically closed field of characteristic \( p > 2 \) and \( M \) a faithful \( L \)-module of dimension \( p - 1 \). Then \( L \) is either a direct sum of classical algebras or the Witt algebra.

The representations of the Witt algebra are well known [2]. The possible dimensions of the representation modules are \( p - 1 \) or \( p^r \) (\( r \geq 0 \)). Semisimple Lie algebras having a representation of lower dimension than \( p - 1 \)

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(but $>1$) are direct sums of simple classical Lie algebras as is shown by J. B. Jacobs [3].

II. Filtrations. We use the notation $(xyz) = (xy)z$, $xL^i := \langle xy_1 \cdots y_j \rangle$, $0 \leq j \leq i$, $xL^0 := \langle x \rangle$, where $\langle \rangle$ means the linear span. Let $L$ be a simple strongly degenerate Lie algebra. A filtration defined by the proper subalgebra $L_0$ we denote by $\mathcal{F}(L_0)$ and the length of this filtration by $\delta \mathcal{F}(L_0)$. $\delta(L)$ means the maximum of $\delta \mathcal{F}(L_0)$ for all proper subalgebras. This maximum exists, for $L$ is simple and finite dimensional.

Kostrikin proved the existence of an element $c \neq 0$ for which

\[
(K_1 R_{x_1} \cdots R_{x_{p-4}} R_c = 0, \quad (K_2 R_{x_1} \cdots R_{x_{p-3}} R_c R_{y_1} \cdots R_{y_{p-3}} R_c = 0)
\]

hold [6].

**PROPOSITION 1.**  (a) $\delta(L) \geq p-3$.

(b) If $\delta(L) = k$, then there exists an element $c \in L$ with $R_c R_{x_1} \cdots R_{x_i} R_c = 0$ for all $i \leq k-1$ if $k$ is even and $i \leq k-2$ if $k$ is odd.

**PROOF.**  (a) Using an element $c$ satisfying (*) we define $L_0 := \ker R_c$ This is a proper subalgebra containing $(cL^{p-3})$; so $\delta \mathcal{F}(L_0) \geq p-3$.

(b) Let $\mathcal{F}(L_0)$ be a filtration of length $k$. If $k = 2m + 1$ then $(cx_0 \cdots x_i) e L_1$ for all $i \leq k-2$ and any $c \in L_k$. Thus $(cx_0 \cdots x_i c) = 0$ and $R_c R_{x_1} \cdots R_{x_i} R_c = 0$. If $k = 2m$, $c \in L_k$, then

\[
((cx_1 \cdots x_i)(cy_1 \cdots y_j)) \in L_{k-i+k-j} = (0) \quad \text{if } i + j < k.
\]

Therefore it holds that

\[
0 = ((cx_1 \cdots x_m)(cx_1 \cdots x_m)) = (-1)^m(cx_1^2 \cdots x_m^2 c).
\]

By linearization it follows that $(cy_1 \cdots y_{2m} c) = 0$ for all $y_i \in L$.

**PROPOSITION 2.**  There exists a nilpotent filtration of length $\geq p-3$.

**PROOF.**  Using an element $c$ satisfying (*) we define $L_0 := (cL^{p-3}) \subset \ker R_c$.

$L_0$ is a proper subalgebra, and $\delta \mathcal{F}(L_0) \geq p-3$. We prove that $L_0$ acts nilpotently on $L$:

\[
R(cx_1^{p-3}) \cdots R(cx_n^{p-3}) = \sum \alpha(i_1, \cdots, i_n)R(x_1)^{i_1}R(c)R(x_2)^{p-3-i_1}R(x_3)^{i_2}R(c) \cdots R(c)R(x_n)^{p-3-i_n}.
\]

If $p-3 - i_j + i_{j+1} \leq p-3$ and $p-3 - i_{j+1} - i_{j+2} \leq p-3$ for some $j$ then this summand equals 0 by (*). Thus assume $i_{j+1} > i_j$ for all $j = 1, \cdots, n-2$. It
follows that \( i_{q+1} \geq r \) which is impossible if \( n > 2(p - 2) \). By linearization the result follows.

The natural number \( \delta(L) \) is closely connected with the concept of height of a Lie algebra mentioned by Kostrikin [6]. The height \( \Delta(L, x) \) of \( L \) with respect to \( x \) is defined to be the minimal index \( n \) for which \( (xL^n) = L \). Then \( \Delta(L) \), the maximum of \( \Delta(x, L) \) over all \( x \) in \( L \), he called the height of the simple Lie algebra \( L \). We get a filtration of length \( \Delta(L, x) - 2 \) by the following construction.

Define \( L_0 = \{ z \in L \mid (zL) \subset xL^{\Delta(L, x) - 1} \} \). If \( z_1, z_2 \in L_0 \), then

\[
((z_1z_2)u) = ((z_1u)z_2) + (z_1(z_2u)) \not\in xL^{\Delta(L, x) - 1}.
\]

Thus \( L_0 \) is a proper subalgebra defining a filtration of length \( \Delta(L, x) - 2 \). If, on the other hand, \( \delta_S(L_0) = k \) and \( e \in L_k \), then \( (eL^k) \subset L_0 \) and \( \Delta(L, e) \geq k + 1 \) holds. This proves

**Proposition 3.** \( \Delta(L) \geq \delta(L) + 1 \geq \Delta(L) - 1 \).

\( \Delta(L) = \delta(L) + 1 \) (equal to \( p - 1 \)) holds for the Witt algebra, \( \Delta(L) = \delta(L) + 2 \) (equal to \( 2p - 2 \)) for the Block algebra \( L(G_0, 0, f) \) of dimension \( p^2 - 1 \). Kostrikin stated [6] that there exists a fixed \( \rho \) such that \( \Delta(L) \leq \rho \) for all classical simple Lie algebras. The main result [7] of Kostrikin and the extensions by J. B. Jacobs [3] and the author [9] can be combined to read:

\( \delta(L) = 1 \) if and only if \( L \) is classical, provided the ground field is algebraically closed (thus \( \Delta(L) \leq 3 \) in this case).

It is an open question whether there exist Lie algebras for which \( \delta(L) = 0 \) holds. We need a characterization of these algebras obtained in [9].

**Proposition 4.** Let \( L \) be a simple Lie algebra of characteristic \( p > 3 \). Then \( \delta(L) = 0 \) if and only if \( R_x^{p-1} \neq 0 \) for all \( x \in L \).

**III. Proof of the theorem.** The proof is done by several lemmas. If \( p = 3 \) an elementary computation shows that \( L \) is the three-dimensional simple Lie algebra. So we assume \( p > 3 \). An argument of J. B. Jacobs [3] based upon a result of Block on semisimple Lie algebras [1] ensures

\[
\bigoplus \text{der } S_i \supset L \supset \bigoplus \text{ad } S_i
\]

where \( S_i \) are simple Lie algebras and \( \text{der } S_i \) denotes the derivation algebra of \( S_i \). Let \( S_i := S \). We claim \( S \) possesses a faithful irreducible representation module \( N \) of dimension \( \leq p - 1 \). If not, there exists \( m \in M \) such that \( mS = (0) \). \( M/Km \) is a faithful module of lower dimension. By induction it follows that \( S \) is nilpotent, a contradiction. By \( T \) we denote the representation \( S \to \text{End}(N) \).

**Lemma 1.** For \( y \) in \( L \), \( T(y) \) is nilpotent if and only if \( R_y \) is nilpotent. If either (and then both) is nilpotent, then \( R_x^p = T(y)^{p-1} = 0 \).
PROOF. $T(y)$ nilpotent implies $T(y)^{p-1}=0$. It follows that $uT((zy^n)) = uT(z)T(y)^{p-1}uT(y)^{p-1}T(z)=0$ for all $z \in L$, $u \in N$, whence $R_y^n=0$.

Let $R(y)$ be nilpotent, that is $(Ly^n)=0$. Then $T^p(y)=\lambda \text{Id}_N$ because $N$ is irreducible. $T(y)$ has only one eigenvalue $\mu$ and $0=\text{tr}(T(y))=\mu \dim N$. Thus $\lambda=\mu^n=0$, $T(y)$ is nilpotent. $T^{p-1}(y)=0$ holds for $\dim N \leq p-1$.

(1) Now we assume $\delta(S)=0$.

LEMMA 2. (a) Let $T(y)$ be nilpotent. Then it follows that $NT^k((ay^{p-1})) \subset NT^k(y)$ for all $a \in S$, $k=1, \ldots, p-1$.

(b) If $T^{p-2}(y)=0$ then $y=0$.

PROOF. $T^k(ay^{p-1})=\sum x(((i_1, \ldots, i_{k+1}))T(y)^iT(a)T(y)^i \cdots T(a)T(y)^i_{k+1}$. If $i_j \geq p-1$ for some $i$ this term vanishes. So we assume $i_j \leq p-2$ for all $j$. We have $k(p-1)=i_{k+1}+\sum_{j=1}^k i_j \leq (p-2)k+i_{k+1}$, that is $i_{k+1} \geq k$. If $T^{p-2}(y)=0$ we can assume $i_j \leq p-3$. It follows that $(k=(p-1)/2)$

$$\left(\frac{p-1}{2}\right)(p-1) = i_{(p+1)/2} + \sum_{j=1}^{(p-1)/2} i_j \leq i_{(p+1)/2} + (p-3)\left(\frac{p-1}{2}\right),$$

that is, $i_{(p+1)/2} \geq p-1$, and $T^{(p-1)/2}((ay^{p-1}))=0$ for all $a \in S$. Now $R_y^{p-1} \neq 0$ if $y \neq 0$ (for $\delta(L)=0$). There exists $a \in S$ such that $z:=ay^{p-1} \neq 0$. We have

$$uT((bz^{p-1})) = \sum_{i=0}^{p-1} uT(z)T(b)T^{p-1-i}(z) = 0 \quad \text{for all} \quad u \in N, \quad b \in S.$$ 

From this it follows that $(bz^{p-1})=0$ for all $b \in S$, so $z=0$. This contradiction proves the lemma.

Now let $H$ be a Cartan subalgebra of $S$ and $S=\sum S_x$, $N=\sum N_x$ decompositions with respect to $R_H$ and $T(H)$.

If $x \neq 0$ is any root and $x \in S_x$, by Lemma 2, $NT^{p-3}(x) \neq 0$. There exists $u \in N$ such that $uT^{p-3}(x) \neq 0$. Then the $N_{\beta+i\alpha}$ ($0 \leq i \leq p-2$) are $(p-1)$ different weight spaces. It follows that $\dim N_{\beta+i\alpha}=1$. Let $h \in H$. Then

$$0 = \text{tr}(T(h)) = \sum_{i=0}^{p-2} (\beta+i\alpha)(h) = (p-1)\beta(h) + \alpha(h),$$

whence $\alpha=\beta$.

We proved $N_xT(x) \neq 0$ for all $x \in S_x$ and roots $x \neq 0$. But then $\dim S_x=1$ is true because all weight spaces of $N$ are one dimensional. If $\dim H \neq 1$ there exists $h \in H$ such that $\alpha(h)=0$. It then holds that $T(h)=0$, a contradiction. Therefore $\dim S=\dim H=1$ and $S$ is the Witt algebra, $\delta(S) \neq 0$.

(2) $S$ is classical if $\delta(S)=p$ holds and $S$ is the Witt algebra, $\delta(S) \neq 0$.

If $p=5$ a detailed computation shows that $S$ has nondegenerate trace $\text{tr}(T(x)T(y))$, and thus is classical. From $\dim N \leq p$ and [8] follows that $S$ has nondegenerate trace form in each case. This implies der $S=\text{ad} S$. 

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(3) We assume $\delta(S) = t > 1$. By Proposition 2 there exists a nilpotent filtration $\mathcal{F}(S_0)$ of length $p-3$ if $t = p-3$ and of length $t-1$ if $t > p-3$. From Lemma 1 it follows that $T(S_0)$ is a Lie algebra of nilpotent transformations on $N$. There exists $m \in N$, $m \neq 0$, for which $mT(S_0) = 0$. We define an ascending chain of vector spaces by

$$N_0 := Km, \quad N_i := \langle mT(x_1) \cdots T(x_i) \rangle, \quad 0 \leq j \leq i, \quad x_i \in S.$$

$N$ is irreducible, so $N_k = N_{k+1} = N$ for some $k$.

**Lemma 3.**  
(a) $N_i T(S_{p-3-i}) = (0)$ if $i + j < p - 3$.  
(b) $N_k \neq N$ if $k < p - 2$.  
(c) dim $N_i / N_{i+1} = 1$, $0 \leq i \leq p - 2$.  
(d) There exists $x \in S$ such that $N_i = \sum_{j=0}^i KmT^j(x)$.  
(e) $\delta(S_0) \leq p - 3$, $\delta(S) \leq p - 2$.

**Proof.**  
(a) follows easily by induction on $j$.  
(b) $N_kT(S_{p-3}) = (0)$ if $k < p - 2 \Rightarrow N_k \neq N$.  
(c) is a direct consequence of (b) because dim $N \leq p - 1$.  
(d) If $mT^{p-2}(y) \in N_{p-3}$ for all $y \in S$, then by linearization $N_{p-2} \subseteq N_{p-3}$ holds, a contradiction. So there exists $x \in S$ such that $N_{i-1} \neq \langle mT^i(x) \rangle (0 \leq i \leq p - 2)$.  
(e) If $c \in S_{p-2}$, then $mT((cx^i)) = 0$ holds for all $i \leq p - 2$. It follows that $NT(c) = (0)$ and $c = 0$. The construction of $\mathcal{F}(S_0)$ implies $\delta(S) \leq (p - 3) + 1$.

Now we can prove

**Proposition 5.** $S$ is the Witt algebra.

**Proof.** Define $S'_0 := \{ z \in S \mid mT(z) \in Km \}$. $S'_0$ is a proper subalgebra of $S$ containing $S_0$. If $y \in S / S'_0$, then $mT(y) \in N_1 = Km + KmT(x)$. It follows that $mT(y) = \alpha(y)m + \beta(y)mT(x)$ and $y - \beta(y)x \in S'_0$. So $S'_0$ defines a filtration $\mathcal{F}(S'_0)$ of length $\leq p - 2$ for which dim $S / S'_0 = 1$. If $e_1, \cdots, e_r \in S'_0$ are linearly independent modulo $S'_{i+1}$ then $(e_1x, \cdots, (e_rx)$ are linearly independent modulo $S'_i$ by definition of filtrations. This means

$$\dim S'_i / S'_{i+1} \leq \dim S'_{i-1} / S'_i \leq \cdots \leq \dim S / S'_0 = 1.$$

It follows that dim $S = \delta(S_0) + 2 \leq p$. One easily proves from this that $S$ is the Witt algebra.

The equality $\text{der } S = \text{ad } S$ holds for the Witt algebra. As the conclusion of all this we get $L = \bigoplus S_i$, where $S_i$ is either classical or the Witt algebra. Now assume $S_1$ to be the Witt algebra. We show $L' = \bigoplus_{i > 1} S_i = (0)$. There is only one representation of $S_1$ of dimension $p - 1$, but none of lower dimension. If we take a canonical basis $(e_i)$, $-1 \leq i \leq p - 2$, satisfying $e_i e_j = (j-i) e_{i+j}$ (or $= 0$ if $i + j > p - 2$), then there exists $u \in M$ such that $M = \sum_{i=0}^{p-2} KuT^i(e_{-1})$, $uT^{p-1}(e_{-1}) = 0$ and $vT(e_1) = 0$ if and only if $v \in Ku$.
This follows from [2]. From \( uT(L')T(e_i) = uT(e_i)T(L') = 0 \) we have that \( uT(L') \subseteq Ku \) and \( uT(L') = 0 \) (for \( L' = L'L' \)). This leads to \( uT'(e_{-1})T(L') = uT(L')T'(e_{-1}) = 0 \) for all \( i \) and \( L' = 0 \). Q.E.D.

REFERENCES


MATHEMATISCHES SEMINAR DER UNIVERSITÄT, 2 HAMBURG 13, ROTHENBAUMCHAUSSEE 67/69, FEDERAL REPUBLIC OF GERMANY