REAL HOMOGENEOUS ALGEBRAS

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ABSTRACT. Let \((A, \mu)\) be a finite dimensional real algebra (not necessarily associative) with multiplication \(\mu \neq 0\). Assuming that \(\text{Aut}(A)\) is transitive on one-dimensional subspaces we determine all such algebras. There are up to isomorphism only four such algebras, one in each of the dimensions 1, 3, 6, 7.

Introduction. For the terminology we refer to Bourbaki [3]. All the algebras considered in this paper are assumed to be finite dimensional. Let \(A\) be an algebra over a field \(F\), \(\mu : A \otimes A \to A\) its multiplication and \(\text{Aut}(A)\) the group of algebra automorphisms of \(A\). We shall say that \(A\) is homogeneous if \(\text{Aut}(A)\) is transitive on one-dimensional subspaces of \(A\), we shall say that \(A\) is extremely homogeneous if \(\text{Aut}(A)\) is transitive on \(A\setminus\{0\}\). Kostrikin [7] has shown that if \(\text{char } F \neq 2\), \(A\) extremely homogeneous and \(\mu \neq 0\), then \(F\) must be a finite field. On the other hand Shult [11] has shown that if \(A\) is homogeneous, \(F = \text{GF}(q)\), \(q > 2\) and \(\mu \neq 0\) then \(A \cong F\). The case \(F = \text{GF}(2)\) has been considered by Gross [4]. Świerczkowski has shown [13] that when \(F = \mathbb{R}\) (the real field) and \(A\) is a homogeneous Lie algebra with \(\mu \neq 0\) then \(A\) is isomorphic to the Lie algebra of skew-symmetric \(3 \times 3\) real matrices. Many of these results have been improved by Mr. L. Sweet [12]. In particular, he has determined all two-dimensional homogeneous algebras and has shown that there are no nontrivial homogeneous algebras over an algebraically closed field.

In this paper we shall determine all real homogeneous algebras. If \(A\) is an \(F\)-algebra and \(B \subseteq A\) a subspace we define a multiplication in \(B\) by choosing a vector space complement \(C\) for \(B\) in \(A\) and putting

\[
\mu_B(b_1 \otimes b_2) = \pi \mu_A(b_1 \otimes b_2)
\]

where \(\pi : A \to B\) is the projection with kernel \(C\). We say then that \((B, \mu_B)\) is obtained from \((A, \mu_A)\) by truncation. Note that the definition of \(\mu_B\) depends on the choice of \(C\).

Received by the editors July 10, 1972.


Key words and phrases. Homogeneous algebra, Lie group, Lie algebra, linear representation, symmetric and skew-symmetric tensors, weights.

1 This work was supported in part by the NRC-Grant A-5285.

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We can apply this to quaternions $\mathbb{H}$ by choosing $B$ to be the subspace of pure quaternions and $C = \mathbb{R} \cdot 1$. The corresponding truncated algebra $\tilde{\mathbb{H}}$ will be called the algebra of pure quaternions; it is isomorphic to the three-dimensional Lie algebra mentioned above. Similarly, if we take $A = \mathbb{O}$ the algebra of octonions, choose $B$ to be the subspace of pure octonions and $C = \mathbb{R} \cdot 1$ then the corresponding truncated algebra $\tilde{\mathbb{O}}$ will be called the algebra of pure octonions. It is well known that these two algebras are homogeneous.

Let $T = \mathbb{C}^3$ considered as a real vector space and define the multiplication in $T$ as follows: If $x, y \in T$ and

$$ x = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}, \quad y = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}, $$

then

$$ x \cdot y = \begin{pmatrix} \bar{\xi}_2 \bar{\eta}_3 - \bar{\xi}_3 \bar{\eta}_2 \\ \bar{\xi}_3 \bar{\eta}_1 - \bar{\xi}_1 \bar{\eta}_3 \\ \bar{\xi}_1 \bar{\eta}_2 - \bar{\xi}_2 \bar{\eta}_1 \end{pmatrix}. $$

Then $T$ is a homogeneous algebra over reals of dimension 6. The group $SU(3)$ acts on $T$ by matrix multiplication and these multiplications are automorphisms of the algebra $T$.

**Result and proof.** Let $(A, \pi)$ be a homogeneous $F$-algebra, $\mu \neq 0$ and $G = \text{Aut}(A)$. Then $A$ is irreducible as a $G$-module. Let $\Gamma$ be any subgroup of $G$ such that $A$ is an irreducible $\Gamma$-module. The multiplication $\mu : A \otimes A \to A$ is a homomorphism of $\Gamma$-modules. If $\text{char } F = 0$ then $A \otimes A$ is a semisimple $\Gamma$-module (see [5, p. 85]). Thus $A$ has to be isomorphic as a $\Gamma$-module to a direct summand of $A \otimes A$.

**Theorem.** If $(A, \mu)$ is a real homogeneous algebra and $\mu \neq 0$ then it is isomorphic to $\mathbb{R}, \mathbb{H}, T$ or $\mathbb{O}$.

**Proof.** If $\dim A = n = 1$ this is clear. Let $n \geq 2$ and identify the sphere $S^{n-1}$ with the manifold of oriented one-dimensional subspaces of $A$. We have the canonical maps $A \setminus \{0\} \to S^{n-1} \to P(A)$ where $P(A)$ is the associated projective space. The action of $G$ on $A$ induces an action on $S^{n-1}$ and $P(A)$. The action of $G$ on $P(A)$ is transitive since $A$ is a homogeneous algebra. Since $G$ is a real Lie group its identity component $G_0$ is also transitive on $P(A)$ and $S^{n-1}$.

If $n \geq 3$ then $S^{n-1}$ is simply connected and, by a result of Montgomery [9], $G_0$ has a compact subgroup $\Gamma$ which is also transitive on $S^{n-1}$. By
taking a subgroup of $\Gamma$ (if necessary) we can assume in addition that it is a simple Lie group (see [8]). The possible groups $\Gamma$ which satisfy all these conditions have been determined by Montgomery and Samelson [8] and Borel [1] and [2]. They are as follows:

- $SO(n)$, $SU(m)$ for $n=2m$,
- $SP(m)$ for $n=4m$, $Spin(9)$ for $n=16$,
- $Spin(7)$ for $n=8$, $G_2$ for $n=7$.

Also, the actions of these groups on $S^{n-1}$ are equivalent to the orthogonal actions obtained by embedding $SO(n)$, $SU(m)$, $Sp(m)$ in $SO(n)$ in the usual way; $G_2$ is embedded in $SO(7)$ as the automorphism group of $O$ restricted to $\mathbb{O}$. The embeddings of $Spin(9)$ and $Spin(7)$ are given by the real spin representations $\Delta_9$ and $\Delta_7$. These results have been proved by Poncet [10].

Since $SU(m) \subset SO(n)$ $(n=2m)$ and $Sp(m) \subset SU(2m)$ $(n=4m)$ we can reduce the above list to the following

(i) $n$ odd, $\Gamma = SO(n)$ or $G_2$ (if $n=7$);
(ii) $n=2k$, $k$ odd, $\Gamma = SU(k)$;
(iii) $n=4k$, $\Gamma = Sp(k)$ or $Spin(9)$ (if $n=16$) or $Spin(7)$ (if $n=8$).

We shall now consider each of these possibilities.

(i) If $\Gamma = SO(n)$ then $A \otimes A$ decomposes into direct sum of the symmetric and the skew-symmetric parts. The skew-symmetric part is an irreducible $\Gamma$-module. The symmetric part decomposes into two irreducible summands one of them being the trivial module. By comparison of the dimensions we see that $A$ is not isomorphic as a $\Gamma$-module to any of the summands in $A \otimes A$ if $n \geq 5$. In the case $n=3$, $A$ is isomorphic to the skew-symmetric part of $A \otimes A$. Thus there exists a nonzero $\Gamma$-homomorphism $\mu: A \otimes A \to A$.

Since $A$ is absolutely irreducible as a $\Gamma$-module it follows that $\mu$ is unique up to a scalar factor. This means that the algebra $(A, \mu)$ is unique up to isomorphism. It is clear that this is the algebra of pure quaternions $\mathbb{H}$.

Now, assume that $\Gamma$ is the simple real Lie group $G_2$ and $n=7$. Consider the corresponding representation of the real Lie algebra $G_2$ in $A$. We shall use the notation of Jacobson [6, Chapter VII, Theorem 9] to denote its highest weight by $\lambda_2$. The weights of this representation are all simple and they are

\[ \lambda_2, \quad \lambda_1 - \lambda_2, \quad -\lambda_1 + 2\lambda_2, \quad \lambda_1 - 2\lambda_2, \quad -\lambda_1 + \lambda_2, \quad -\lambda_1, \quad 0. \]

The representation $(\lambda_2) \otimes (\lambda_2)$ decomposes into symmetric and skew-symmetric parts. An irreducible representation of highest weight $2\lambda_2$ is contained in the symmetric part of $(\lambda_2) \otimes (\lambda_2)$. The dimension of $(2\lambda_2)$ is 27 and the dimension of the symmetric part of $(\lambda_2) \otimes (\lambda_2)$ is 28. Hence, the symmetric part of $(\lambda_2) \otimes (\lambda_2)$ decomposes as $(2\lambda_2) + (0)$ where $(0)$ stands for the trivial irreducible representation (of dimension 1).
Similarly, by analyzing the weights of the skew-symmetric part of \((\lambda_2) \otimes (\lambda_2)\) we find that it decomposes as \((\lambda_1) \oplus (\lambda_3)\).

Thus we have \((\lambda_2) \otimes (\lambda_2) = (2\lambda_2) \oplus (0) \oplus (\lambda_1) \oplus (\lambda_3)\).

We have an analogous decomposition of \(A \otimes A\) as a \(\Gamma\)-module. Hence there exists a nonzero \(\Gamma\)-homomorphism \(\mu: A \otimes A \to A\). Again \(A\) is absolutely irreducible as a \(\Gamma\)-module and consequently \(\mu\) is unique up to a scalar factor. This means that the algebra \((A, \mu)\) is unique up to isomorphism.

Of course, this algebra is isomorphic to \(O\) the algebra of pure octonions.

(ii) Now let \(\Gamma = SU(k), n = 2k\). Then \(A\) can be equipped with a complex structure so that it becomes complex \(\Gamma\)-module of complex dimension \(k\).

We shall denote this complex \(\Gamma\)-module by \(A_C\). Moreover, there is a positive definite hermitian form \(\langle x, y \rangle\) on \(A_C\) which is preserved by \(\Gamma\).

We shall also consider \(A_C\) as a complex module over the Lie algebra \(L = su(k)\) of \(SU(k)\). Then of course, \(A_C\) can be considered also as a module over the complexified Lie algebra \(L_C = sl(k, \mathbb{C})\). The real \(L\)-module \(A\) is obtained from \(A_C\) by restriction of scalars.

We are interested in analyzing the \(L\)-module \(A \otimes_R A\). By complexification we get the \(L_C\)-module \((C \otimes_R A) \otimes_C (C \otimes_R A)\). We have \(C \otimes_R A = M \oplus N\), where

\[
M = \{1 \otimes x - i \otimes ix \mid x \in A\}, \quad N = \{1 \otimes x + i \otimes ix \mid x \in A\},
\]

are \(L_C\)-submodules. The map

\[A_C \to M, \quad x \mapsto 1 \otimes x - i \otimes ix,\]

is an isomorphism of \(L_C\)-modules. The bilinear form \(\phi\) on \(C \otimes_R A\) which sends \((\alpha \otimes x, \beta \otimes y) \mapsto \alpha \beta \langle x, y \rangle\) is \(L_C\)-invariant and nondegenerate. If we agree that \(\langle x, y \rangle\) is linear in the first variable then the restriction of \(\phi\) to \(M \times N \to C\) is also nondegenerate since

\[\phi(1 \otimes x - i \otimes ix, 1 \otimes y + i \otimes i y) = 4\langle x, y \rangle.\]

Hence the \(L_C\)-module \(N\) is isomorphic to the contragredient of \(M\), i.e., \(C \otimes_R A \cong A_C \oplus A_C^*\). It follows that the complexification of \(A \otimes_R A\) is isomorphic to

\[(A_C \otimes_C A_C) \oplus (A_C^* \otimes_C A_C^*) \oplus 2(A_C \otimes_C A_C^*)\]

as an \(L_C\)-module.

The highest weight of \(A_C\) is \(\lambda_1\) and that of \(A_C^*\) is \(\lambda_{k-1}\). Here we use again the notation of Jacobson [6, Chapter VII, Theorem 6]. By an easy computation we find that

\[(\lambda_1) \otimes (\lambda_1) = (2\lambda_1) \oplus (\lambda_2),\]

\[(\lambda_{k-1}) \otimes (\lambda_{k-1}) = (2\lambda_{k-1}) \oplus (\lambda_{k-2}),\]

\[(\lambda_1) \otimes (\lambda_{k-1}) = (\lambda_1 + \lambda_{k-1}) \oplus (0),\]
where $(\lambda)$ denotes the isomorphism class of an irreducible $L_{c}$-module of highest weight $\lambda$.

If $k>3$ then we see from these decompositions that $A$ cannot be a direct summand of $A \otimes A$.

When $k=3$ we see that $A \otimes A$ contains $A$ as a direct summand with multiplicity one. Hence, there exists a nonzero $\Gamma$-homomorphism $\mu: A \otimes A \to A$ and the algebra $(A, \mu)$ is homogeneous and has dimension 6. The endomorphism ring of the $\Gamma$-module $A$ is the complex field $C$ and it follows easily that $(A, \mu)$ must be isomorphic to the algebra $T$ described in the introduction.

(iii) In all these cases $-I \in \Gamma$ and $(-I) \otimes (-I)$ is the identity map on $A \otimes A$. Hence $A \otimes A$ is not faithful as a $\Gamma$-module and $A$ cannot be isomorphic to a direct summand of $A \otimes A$.

It remains to consider the case $n=2$. Assume that $A \otimes A = V_{1} \oplus V_{2} \oplus V_{3}$ where $V_{1}$ is the skew-symmetric part, $V_{2} \oplus V_{3}$ the symmetric part and $V_{3} \cong A$. Let $\pi_{i}: A \otimes A \to V_{i}$ $(i=1, 2, 3)$ be the associated projections. If $e_{1}, e_{2}$ is a basis of $A$ then $e_{1} = e_{1} \otimes e_{2} - e_{2} \otimes e_{1}$ spans $V_{1}$. Define the bilinear form $f: A \otimes A \to R$ by $\pi_{1}(x \otimes y) = f(x, y)v_{1}$. Then $f$ is skew-symmetric and $f(\sigma x, y) = (\det \sigma)f(x, y)$ for all $\sigma \in \text{GL}(A)$. Similarly, define a bilinear form $g: A \otimes A \to R$ by $\pi_{2}(x \otimes y) = g(x, y)v_{2}$ where $v_{2} \in V_{2}$ is a fixed nonzero tensor. Then $g$ is symmetric and we must have

$$g(\sigma x, y) = (\det \sigma)^{2}g(x, y)$$

for $\sigma \in G_{0}$ and $x, y \in A$. The factor $(\det \sigma)^{2}$ is obtained by considering the determinants of the restrictions of $\sigma \otimes \sigma$ to $V_{i}$ for $i=1, 2, 3$. Both $f$ and $g$ are nondegenerate and we can write $g(x, y) = f(\rho x, y)$ for some fixed $\rho \in \text{GL}(A)$. Then we must have

$$f(\rho \sigma x, y) = (\det \rho)^{2}f(\rho x, y), \quad f(\sigma^{-1} \rho \sigma^{-1} x, y) = (\det \sigma)f(x, y).$$

Thus $\pi_{1} \circ (\sigma^{-1} \rho \sigma^{-1} \otimes I) = (\det \sigma)\pi_{1}$ and consequently $\sigma^{-1} \rho \sigma^{-1}$ must be a scalar transformation, i.e., $\sigma^{-1} \rho \sigma^{-1} = \pm I$. Since $G_{0}$ is connected we have $\rho \sigma = \sigma \rho$ for all $\sigma \in G_{0}$. It follows from a formula above that $\det \sigma = 1$ for $\sigma \in G_{0}$. Hence $g$ is a $G_{0}$-invariant nondegenerate symmetric form. Since $G_{0}$ is transitive on $P(A)$ we conclude that $g$ must be definite, say positive definite. By considering $A$ as a Euclidean space with $g$ as an inner product it is clear that $G_{0} = \text{SO}(2)$.

But now $A \otimes A$ is not a faithful $G_{0}$-module because $-I \in \text{SO}(2)$ and we have a contradiction. This completes the proof.

**Remark.** We could have dismissed the case $n=2$ because it has been shown by L. Sweet that the two-dimensional homogeneous algebras exist only over $GF(2)$.
I am very grateful to the editor and another referee for their critical comments on the first version of this paper.

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