FUNCTIONS POSSESSING RESTRICTED MEAN VALUE PROPERTIES¹

DAVID HEATH

Abstract. A real-valued function $f$ defined on an open subset of $\mathbb{R}^n$ is said to have the restricted mean value property with respect to balls (spheres) if, for each point $x$ in the set, there exists a ball (sphere) with center $x$ and radius $r(x)$ such that the average value of $f$ over the ball (sphere) is equal to $f(x)$. If $f$ is harmonic then it has the restricted mean value property. Here new conditions for the converse implication are given.

1. Introduction. Suppose $D \subseteq \mathbb{R}^N$ is open and $r : D \rightarrow \mathbb{R}$ is positive and $r(x) \leq d(x, D^c)$, the distance from $x$ to the complement of $D$. Let $B(x, r)$ denote the open ball of radius $r$ about $x$, and $S(x, r)$ the corresponding sphere. If $f : D \rightarrow \mathbb{R}$ satisfies

\begin{equation}
(1.1) \quad f(x) = \frac{1}{\mu(B(x, r(x)))} \int_{B(x, r(x))} f(y) \mu(dy), \quad \forall x \in D,
\end{equation}

where $\mu$ is Lebesgue measure in $\mathbb{R}^N$, we say that $f$ possesses the restricted mean value property on balls (in $D$ with respect to $r$). If $f$ satisfies

\begin{equation}
(1.2) \quad f(x) = \frac{1}{v(S(x, r(x)))} \int_{S(x, r(x))} f(y) v(dy), \quad \forall x \in D,
\end{equation}

where $v(dy)$ is the element of surface area on $S(x, r(x))$, we say that $f$ possesses the restricted mean value property on spheres. Of course every harmonic $f$ possesses the restricted mean value property on balls and spheres for arbitrary $r$; we are concerned here with partial converses to this fact.

Several partial converses are known: Courant and Hilbert [4] prove that if $D$ is relatively compact and regular for the Dirichlet problem and $f$ is continuous on $\overline{D}$ and possesses the restricted mean value property on spheres, then $f$ is harmonic. Föllmer [6], using probabilistic techniques, proved a similar result (which he also generalized) requiring $r$ to be Borel measurable, but his only assumption on $D$ is that Brownian paths leave $D$

¹ Research supported in part by the National Science Foundation.

Received by the editors February 5, 1973.

AMS (MOS) subject classifications (1970). Primary 60J45, 31B05.

Key words and phrases. Harmonic function, mean value property, Markov process.

© American Mathematical Society 1973

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
almost surely and \( f \) need be continuous only almost everywhere on \( \partial D \) and on all of \( D \).

Baxter [2], extending a result of Ackoglu and Sharpe [1], showed that if \( D \) is a compact \( C^1 \) manifold with boundary and \( r \) is measurable and satisfies \( r(x) \geq \epsilon_0 \frac{d(x, D^c)}{d(x, D)} \), then any bounded measurable \( f \) possessing the restricted mean value property on balls is harmonic.

More recently, Veech [7] and [8] showed that if \( N = 2 \), and in [9], if \( N \geq 1 \), \( D \) is a relatively compact Lipschitz domain, and \( r \) is bounded away from zero on compact subsets of \( D \), then every Lebesgue measurable function whose absolute value is dominated by some harmonic function on \( D \) and which possesses the restricted mean value property on balls is harmonic.

In this paper we shall prove the following two theorems:

**Theorem 1.** Suppose \( f \) is bounded, \( D \) is a proper subset of \( \mathbb{R}^N \) and \( r \) satisfies
\[
ed(x, D^c) < r(x) < (1 - \epsilon)d(x, D^c)
\]
for some \( \epsilon > 0 \). If \( f \) possesses the restricted mean value property on balls, then \( f \) is harmonic.

**Theorem 2.** Suppose \( f \) is bounded, \( N \geq 2 \), and \( r \) satisfies
\[
|r(x) - r(y)| < (1 - \epsilon_0)|x - y|
\]
for some \( \epsilon_0 > 0 \). If \( f \) possesses the restricted mean value property on spheres then \( f \) is harmonic.

2. Preliminary results. Suppose \((\Omega^0, x^0, \mathcal{F}_t^0, \theta_t^0, P_x^0, x \in \mathbb{R}^N)\) is standard \( N \)-dimensional Brownian motion. Suppose \((\rho^1_n)\) is a sequence of independent, identically distributed random variables on \((\Omega^1, \mathcal{F}^1, P^1)\) with distribution function \( G \) defined by
\[
G(x) = \begin{cases} 
0 & \text{if } x \leq 0, \\
x^N & \text{if } 0 \leq x \leq 1, \\
1 & \text{if } 1 \leq x.
\end{cases}
\]

Let \( \Omega = \Omega^0 \times \Omega^1 \), \( P_x = P_x^0 \times P^1 \), \( \mathcal{F}_t = \mathcal{F}_t^0 \times \mathcal{F}^1 \), and define \( \theta_t \) by
\[
\theta_t(\omega^0, \omega^1) = (\theta_t^0(\omega^0), \omega^1).
\]

Extend \( x^0 \) and \( \rho^1_n \) by \( x(t, (\omega^0, \omega^1)) = x^0(t, \omega^0) \) and \( \rho_n(\omega^0, \omega^1) = \rho^1_n(\omega^1) \).

Given \( r: D \to \mathbb{R} \), measurable, with \( 0 < r(x) < d(x, D^c) \) we define two sequences of stopping times as follows:
\[
\tau_0(\omega) = 0,
\tau_{n+1}(\omega) = \inf\{t: t \geq \tau_n(\omega), |x(t, \omega) - x(\tau_n(\omega), \omega)| \geq \rho_{n+1}(\omega) \cdot r(x(\tau_n(\omega), \omega))\}
\]
and
\[ 
\sigma_0(\omega) = 0,
\]
\[ 
\sigma_{n+1}(\omega) = \inf\{t: t \geq \sigma_n(\omega), |x(t, \omega) - x(\sigma_n(\omega), \omega)| \geq r(x(\sigma_n(\omega), \omega))\}.
\]
Let \( \mathcal{G}_n \) and \( \mathcal{H}_n \) denote the minimal \( \sigma \)-fields for the sequences \( x(\tau_n) \) and \( x(\sigma_n) \) respectively. Then clearly \( (x(\tau_n), \mathcal{G}_n) \) is a Markov chain and the conditional distribution of \( x(\tau_{n+1}) \) given \( x(\tau_n) \) is uniform on \( B(x(\tau_n), r(x(\tau_n))) \); a similar statement holds for \( x(\sigma_{n+1}) \) and \( S(x(\sigma_n), r(x(\sigma_n))) \).

Define \( \tau \) by \( \tau(\omega) = \inf\{t \geq 0: x(t, \omega) \notin D\} \) if this set is nonempty; otherwise set \( \tau(\omega) = +\infty \).

**Lemma 1.** If \( r \) satisfies (1.3) then \( \lim_{n \to \infty} \tau_n(\omega) = \tau(\omega) \) \( P_x \)-a.s. for every \( x \in D \). If \( r \) satisfies (1.4) then \( \lim_{n \to \infty} \sigma_n(\omega) = \tau(\omega) \) \( P_x \)-a.s. for every \( x \in D \).

**Proof.** We prove only the first statement: Let \( \tau'(\omega) = \lim_{n \to \infty} \tau_n(\omega) \); clearly \( \tau' \leq \tau \) since \( \tau_n \leq \tau \). Suppose \( \tau'(\omega) < \infty \). Then \( x(\tau'(\omega), \omega) = \lim_{n \to \infty} x(\tau_n(\omega), \omega) \). If \( x(\tau'(\omega), \omega) \in D \) then \( d(x(\tau'(\omega), \omega), D^c) > 0 \), so by (1.3),
\[ 
\liminf_{n \to \infty} r(x(\tau_n(\omega), \omega)) > 0.
\]
But then
\[ 
\limsup_{n \to \infty} \rho_{n+1}(\omega) = \limsup_{n \to \infty} \frac{|x(\tau_{n+1}(\omega), \omega) - x(\tau_n(\omega), \omega)|}{r(x(\tau_n(\omega), \omega))} = 0;
\]
this event has probability zero so \( \tau'(\omega) < \infty \Rightarrow x(\tau'(\omega), \omega) \notin D \Rightarrow \tau(\omega) \leq \tau'(\omega) \).

**Remark.** We clearly have
\[ 
\bigcup_{t=m}^{\infty} B(x(\tau_t(\omega), \omega), r(x(\tau_t(\omega), \omega))) \supseteq \{y: y = x(t, \omega) \text{ for some } t \in [\tau_t(\omega), \tau(\omega))\}.
\]
This inclusion remains valid if all \( \tau_t \)'s are replaced by \( \sigma_t \)'s.

**Lemma 2.** Suppose \( f \) is bounded and measurable on \( D \) and satisfies (1.1) or (1.2). There is a bounded measurable function \( F \) on \( \Omega \) such that \( f(x) = E_x(F(\omega)) \forall x \in D \).

**Proof.** If \( f \) satisfies (1.1) then \( f(x(\tau_n(\omega), \omega)) \) is a bounded martingale, so by the martingale convergence theorem it converges to some function \( F \) \( P_x \)-a.s. for every \( x \in D \) and the above equation holds. If \( f \) satisfies (1.2), replace \( \tau_n \) by \( \sigma_n \) in the above argument.
For a proof of the following result, see Chung [3]:

**Lemma 3.** Suppose \((M, \mathfrak{A}, m)\) is a probability space and \((\mathfrak{A}_n)\) is an increasing sequence of \(\sigma\)-fields with \(\mathfrak{A}_n \subseteq \mathfrak{A}\). Let \(\mathfrak{A}_\infty = \bigcap_{n=1}^{\infty} \mathfrak{A}_n\), and suppose \(A_i \in \mathfrak{A}_\infty\). Then

\[
\lim_{n \to \infty} P\left( \bigcap_{i=n}^{\infty} A_i \mid \mathfrak{A}_n \right) = 1 \bigcap_{i=1}^{\infty} A_i.
\]

**Lemma 4.** Under the hypotheses of Theorem 2, there exists a \(\delta > 0\) such that if \(x_i, A_i, i=1,2\), satisfy \(x_i \in D, A_i\) measurable and \(A_i \subseteq D, P^2(x_i, A_i) > 1 - \delta\) and \(x_2 \in B(x_1, r(x_1))\) then \(A_1 \cap A_2 \neq \emptyset\).

**Proof.** Define \(A(x, r_1, r_2) = \{y : r_1 < |y - x| < r_2\}\), let \(\varepsilon' = \varepsilon_0 / 4\) and let \(x\) denote either \(x_1\) or \(x_2\). Suppose \(y \in A(x, (1 - \varepsilon')r(x), (1 + \varepsilon')r(x))\) and \(a\) is chosen small enough that \(B(y, a) \subseteq A(x, (1 - \varepsilon')r(x), (1 + \varepsilon')r(x))\). We shall show that

\[
P^2(x, B(y, a)) \geq c(\varepsilon_0)a^{N-1}/(r(x))^N.
\]

To this end, let \(C = \{z \in S(x, r(x)) : |r(z) - z - y| \leq a/2\}\). (Note that \(|r(z) - z - y|\) is the distance by which the sphere \(S(z, r(z))\) misses the point \(y\).) Simple estimates of the central angle of the part of the sphere \(S(z, r(z))\) which is contained in \(B(y, a)\) give

\[
z \in C \Rightarrow P^1(z, B(y, a)) \geq c_1(\varepsilon_0)a^{N-1}/(r(x))^N - 1.
\]

Using the inequalities on (and also the continuity of) \(r\), one shows easily that \(x + r(x)((y - x)/|y - x|)\) (respectively \(x - r(x)((y - x)/|y - x|)\)) are not in \(C\) because \(r\) is too big (small) there, and hence the intersection of \(C\) with each great circle on \(S(x, r(x))\) through \(x + r(x)((y - x)/|y - x|)\) is nonvoid; moreover, the measure of this intersection and the distance of the intersection from \(x \pm r(x)((y - x)/|y - x|)\) are easily shown to be bounded below appropriately so that \(P^1(x, C) \geq c_2(\varepsilon_0)a/r(x)\). (2.1) now follows easily. Notice that (2.1) implies that for any measurable \(E \subseteq A(x, (1 - \varepsilon')r(x), (1 + \varepsilon')r(x))\)

we have

\[
P^2(x, E) \geq c(\varepsilon_0)\mu(E)/(r(x))^N.
\]

To complete the proof, set

\[
I = A(x_1, (1 - \varepsilon')r(x_1), (1 + \varepsilon')r(x_1)) \cap A(x_2, (1 - \varepsilon')r(x_2), (1 + \varepsilon')r(x_2)).
\]

An elementary computation (again using the properties of \(r\)), using the
fact that \( x_2 \in B(x_1, r(x_1)) \), shows that

\[
(2.3) \quad \mu(I) \geq k(e_0) r(x_1)^N r(x_2) \geq k(e_0) r(x_i)^N, \quad i = 1, 2.
\]

We now select \( \delta \) so that under the hypotheses of the theorem we can conclude \( \mu(I \cap A_1^j) + \mu(I \cap A_2^j) < \mu(I) \) so that \( A_1 \cap A_2 \) must be nonvoid.

From (2.2) we obtain \( \mu(I \cap A_1^j) \leq (r(x_i))^N P^2(x_i, I \cap A_1^j)/c(e_0) \), so that by (2.3)

\[
\mu(I \cap A_1^j)/\mu(I) \leq P^2(x_i, I \cap A_1^j)/c(e_0) k(e_0).
\]

Hence if \( \delta = c(e_0) k(e_0)/4 \) we obtain

\[
\mu(I \cap A_1^j) + \mu(I \cap A_2^j) \leq \mu(I)/2 < \mu(I)
\]
as desired. This completes the proof of Lemma 4.

**Lemma 5.** Suppose \( r \) satisfies (1.3). Let \( P^1(x, A) \) denote the one-step transition probability for the chain \( x(\tau_n(\omega), \omega) \). Then there is a \( \delta > 0 \) such that if \( x_i, A_i, i = 1, 2, \) satisfy \( P^1(x_i, A_i) \geq 1 - \delta, \ x_i \in D, \ A_i \subseteq D, \ x_2 \in B(x_1, r(x_1)) \), then \( A_1 \cap A_2 \neq \emptyset \).

**Proof.** Apply simple geometry and use the techniques of the proof of the previous lemma.

3. Proofs of the theorems.

**Proof of Theorem 1.** By a simple modification of an elementary but clever argument of Veech, [7] or [9], we may and shall assume that \( f \) and \( r \) are Borel measurable. Let \( x_0 \in D \) and \( 0 < r_0 < d(x_0, D^c) \). It suffices to show that

\[
f(x_0) = \frac{1}{v(S(x_0, r_0))} \int_{S(x_0, r_0)} f(y) v(dy).
\]

Let \( y_n(\omega) = x(\tau_n(\omega), \omega) \). By Lemma 2 \( \lim_{n \to \infty} f(y_n) = F \) exists \( P_x \)-almost surely and \( f(x) = E_x(F) \). Let \( \eta(\omega) = \inf\{t > 0 : |x(t, \omega) - x_0| \geq r_0\} \) and \( \theta_\eta \) be the corresponding shift operator. Now by the strong Markov property (see Dynkin [6, p. 100]):

\[
E_{x_0}(F \cdot \theta_\eta \mid \mathcal{F}_n) = E_{x_\eta}(F) \quad P_{x_0 \text{-a.s.}}, \quad x_\eta = x(\eta(\omega), \omega).
\]

It therefore suffices to show that \( F \cdot \theta_\eta = F \) \( P_{x_0 \text{-a.s.}} \), for then

\[
f(x_0) = E_{x_0}(F) = E_{x_0}(E_{x_0}(F \mid \mathcal{F}_n)) = E_{x_0}(E_{x_0}(F \cdot \theta_\eta \mid \mathcal{F}_n))
\]

\[
= E_{x_0}(E_{x_\eta}(F)) = E_{x_0}(f(x_0)) = \frac{1}{v(S(x_0, r_0))} \int_{S(x_0, r_0)} f(y) v(dy).
\]

Now consider the sequence \( z_n(\omega) = y_n(\theta_\eta(\omega)) \). Clearly \( f(z_n) \) is a martingale which converges to \( F \cdot \theta_\eta(\omega) \) \( P_{x_0 \text{-a.s.}} \). Thus we need only show that

\[
\lim_{n \to \infty} f(z_n(\omega)) = \lim_{n \to \infty} f(y_n(\omega))
\]
From the definition of the process $z_n$, an argument similar to the proof of Lemma 1, and the remark following Lemma 1 we see that: For each $m$ there is an $n_0(\omega)$ such that $n \geq n_0(\omega) \Rightarrow z_n(\omega) \in \bigcup_{i=m}^{\infty} B(y_i(\omega), r(y_i(\omega)))$.

Suppose now that $\varepsilon > 0$ and $K > 0$ are prescribed. Let

\begin{align*}
A^\varepsilon_i &= \{ \omega : |f(y_{i+1}(\omega)) - f(y_i(\omega))| < \varepsilon/2 \}, \\
B^\varepsilon_i &= \{ \omega : |f(z_{i+1}(\omega)) - f(z_i(\omega))| < \varepsilon/2 \}.
\end{align*}

Because of the convergence mentioned above, we have

\begin{align*}
P_{x_0} \left( \bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} A^\varepsilon_i \right) &= 1, \\
P_{x_0} \left( \bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} B^\varepsilon_i \right) &= 1,
\end{align*}

so by Lemma 3 and the strong Markov property we have

\begin{align*}
\lim_{m \to \infty} P_{x_0} \left( \bigcap_{i=m}^{\infty} A^\varepsilon_i \mid y_m \right) &= 1, \\
\lim_{m \to \infty} P_{x_0} \left( \bigcap_{i=m}^{\infty} B^\varepsilon_i \mid z_m \right) &= 1 \quad P_{x_0}\text{-a.s.,}
\end{align*}

and hence

\begin{align*}
\lim_{m \to \infty} P_{x_0}(A^\varepsilon_m \mid y_m) &= 1, \\
\lim_{m \to \infty} P_{x_0}(B^\varepsilon_m \mid z_m) &= 1 \quad P_{x_0}\text{-a.s.}
\end{align*}

Now let $C^\varepsilon(x) = \{ y \in D : |f(y) - f(x)| < \varepsilon/2 \}$. It is easy to see that

\begin{align*}
P_{x_0}(A^\varepsilon_m \mid y_m) &= P(y_m, C^\varepsilon(y_m)), \\
P_{x_0}(B^\varepsilon_m \mid z_m) &= P(z_m, C^\varepsilon(z_m))
\end{align*}

and so the right-hand sides converge to one almost surely. Let $\delta$ be as in Lemma 5, and choose $m_0 \geq K$ so large that $m \geq m_0 \Rightarrow P(y_m, C^\varepsilon(y_m)) \geq 1 - \delta$.

Choose $n_0(\omega)$ so large that $n \geq n_0 \Rightarrow z_n(\omega) \in \bigcup_{i=m_0}^{\infty} B(y_m, r(y_m))$, so there is an $x \in D$ with

\begin{align*}
|f(x) - f(y_m)| < \varepsilon/2 \quad \text{and} \\
|f(x) - f(z_n)| < \varepsilon/2
\end{align*}

and thus $|f(y_m) - f(z_n)| < \varepsilon$. Because $K$ was arbitrary, we must have

\begin{align*}
\lim_{n \to \infty} f(y_n) - \lim_{n \to \infty} f(z_n) \leq \varepsilon
\end{align*}

whenever these limits exist, $P_{x_0}$-a.s. Hence $F = F_\eta P_{x_0}$-a.s. and the proof is completed.

**Proof of Theorem 2.** Theorem 2 is proved in the same manner as Theorem 1, using instead the two-step transition probability for the
Markov chains $y'_n$ and $z'_n$ given by

$$y'_n(\omega) = x(\sigma_n(\omega), \omega), \quad z'_n(\omega) = y'_n(\theta_n(\omega)).$$

4. Examples. The first example shows that some restrictions on $r$ are necessary. Let $D$ be the unit disc in $\mathbb{R}^2$, i.e., $\{(x, y): x^2 + y^2 < 1\}$ and define

$$f(x, y) = 1, \quad \text{if } y > 0,$$

$$= 0, \quad \text{if } y = 0,$$

$$= -1, \quad \text{if } y < 0.$$

Let $r(x, y) = \min(|y|, 1 - \|(x, y)\|)$ for $y \neq 0$ and $1 - |x|$ for $y = 0$. Then $f$ satisfies (1.1), is bounded and measurable, but not harmonic.

The second example shows that some restrictions on the growth of $f$ are necessary. Let $D = (0, 1)$ in $\mathbb{R}^1$ and define

$$f(x) = \sum_{n=1}^{\infty} \varphi(n) 1_{\{(n-1)/n, n/(n+1)\}}(x)$$

where $\varphi(k) = 0$, if $k = 1, 2, 3$, and $\varphi(k) = (-1)^k k^2 (k + 1)$, for $k > 3$.

Notice that

$$\int_0^{n/(n+1)} f(x) \, dx = \sum_{k=1}^{n} (-1)^k k^2 (k + 1) \frac{1}{k(k+1)} = \sum_{k=1}^{n} (-1)^k k.$$  

Then if $g(x) = \int_{(n-1)/n}^{x} f(y) \, dy$, lim sup$_{x \to 1} g(x) = +\infty$ and lim inf$_{x \to 1} g(x) = -\infty$ and $g$ is continuous on $[0, 1)$. Moreover for any $x \in (1/2, 1)$ the function

$$h_x(r) = \frac{g(x + r) - g(x - r)}{2r} \quad \text{for } 0 < r < 1 - x$$

is continuous and also has lim sup and lim inf of $+\infty$ and $-\infty$ respectively as $r \to 1 - x$. Thus there are values of $r$ arbitrarily near $1 - x$ for which

$$f(x) = h_x(r).$$

Clearly $r$ can be chosen a measurable function of $x$ to satisfy (4.1). For $0 < x \leq 1/2$ one can define $r(x) \equiv 1/2$. Thus $f$ and $r$ satisfy all of the hypotheses of our theorem except that of boundedness, and $f$ is not harmonic. (A similar example can be constructed in which $f$ is bounded and $D = (-\infty, \infty)$ by setting

$$f(x) = \sum_{n=1}^{\infty} (-1)^n \left(1 - \frac{1}{n}\right) 1_{\{a_n, a_{n+1}\}}(|x|)$$

where $(a_n)$ increases sufficiently rapidly.)
Finally, we remark that any bounded measurable periodic function on $\mathbb{R}^1$ satisfies the hypotheses for Theorem 2 with $r(x)$ equal to the period except that $N=1$. For further examples related to these concepts, see Courant and Hilbert [4].

REFERENCES


9. ———, *A zero-one law for a class of random walks and a converse to Gauss’ mean value theorem* (preprint).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455

Current address: Université de Strasbourg, 67000 Strasbourg, France