COMPACT SUBSETS OF $\mathbb{R}^n$ AND DIMENSION OF THEIR PROJECTIONS

SIBE MARDEŠIĆ

Abstract. In this paper it is proved that a $k$-dimensional closed subset $X \subseteq \mathbb{R}^n$ admits a projection $p$ into one of the coordinate $k$-planes such that $\dim p(X) = k$.

The purpose of this note is to prove the following theorem:

**Theorem.** Let $X \subseteq \mathbb{R}^n$ be a $k$-dimensional compact subset of $\mathbb{R}^n$, $1 \leq k \leq n$. Then there exist $k$ different factors $R_{i_1} = \mathbb{R}, \ldots, R_{i_k} = \mathbb{R}$ of $\mathbb{R}^n$, $1 \leq i_1 < \cdots < i_k \leq n$, such that $\dim p_{i_1 \cdots i_k}(X) = k$, where $p_{i_1 \cdots i_k}$ is the projection $\mathbb{R}^n \to \mathbb{R}_{i_1} \times \cdots \times \mathbb{R}_{i_k}$.

The question of whether the above statement is true was raised by J. D. Lawson in connection with a problem concerning $n$-dimensional topological semilattices on a Peano continuum. I am indebted to J. Nagata for bringing it to my attention.

It is well known that a subset $Y \subseteq \mathbb{R}^k$ is $k$-dimensional if and only if it has nonempty interior, $\text{Int} Y \neq \emptyset$ (see, e.g., [1, Theorem IV.3, p. 44]). Consequently, the theorem can be given the following equivalent form:

**Theorem.** Let $X \subseteq \mathbb{R}^n$ be a $k$-dimensional compact subset of $\mathbb{R}^n$, $1 \leq k \leq n$. Then there exist $1 \leq i_1 < \cdots < i_k \leq n$ such that $\text{Int} p_{i_1 \cdots i_k}(X) \neq \emptyset$ in $\mathbb{R}_{i_1} \times \cdots \times \mathbb{R}_{i_k}$.

We prove the theorem by induction on $n$ using this second form. If $n = 1$, then $k = 1$ and $p_1 : \mathbb{R} \to \mathbb{R}$ is the identity so that $p_1(X) = X$. However, $X \subseteq \mathbb{R}$ must contain a nonempty open set for otherwise $\mathbb{R} \setminus X$ would be dense in $X$, which would imply $\dim X \leq 0$ (inductive dimension), contradicting the assumption $\dim X = 1$.

We now assume that the theorem holds for positive integers $\leq n - 1$, $n \geq 2$, and we prove it for $n$. Let $X \subseteq \mathbb{R}^n$, $\dim X = k$, $k \leq n$. Consider any of

Received by the editors December 13, 1972 and, in revised form, February 16, 1973.


Key words and phrases. Dimension, projection, Euclidean space, Baire category theorem, topological semilattice.

1 This paper has been written while the author was visiting the University of Pittsburgh on leave from the University of Zagreb.

© American Mathematical Society 1973

631
the $n$ factors of $R^n$, say $R_1 = R$, so that $R^n = R_1 \times R^{n-1}$. Let $S_1 \subset R_1$ be the set of all points $\xi_1 \in R_1$ such that

$$\dim(X \cap (\xi_1 \times R^{n-1})) \leq k - 2.$$  

Furthermore, for any $k-1$ different integers $2 \leq i_1 \cdots < i_{k-1} \leq n$, consider all balls $B_{i_1 \cdots i_{k-1}}(q, \varepsilon) \subset R_{i_1} \times \cdots \times R_{i_{k-1}}$ with rational radius $\varepsilon > 0$ and center $q = (q_{i_1}, \cdots , q_{i_{k-1}})$ all of whose coordinates are rational. Let $S_{i_1 \cdots i_{k-1}}(q, \varepsilon)$ be the set of all points $\xi_1 \in R_1$ such that

$$\xi_1 \times B_{i_1 \cdots i_{k-1}}(q, \varepsilon) \subset p_{i_1 \cdots i_{k-1}}(X \cap (\xi_1 \times R^{n-1})).$$

We shall first show that

$$R_1 \mid S_1 \subset \bigcup S_{i_1 \cdots i_{k-1}}(q, \varepsilon),$$

where the union is taken over all sequences $2 \leq i_1 \cdots < i_{k-1} \leq n$ and over all rational $(q, \varepsilon)$ and thus has countably many terms. Indeed, if $\xi_1 \in R_1 \mid S_1$, then $\dim(X \cap (\xi_1 \times R^{n-1})) = l \leq n-1$ is $k-1$ or $k$. By the induction hypothesis, there is a sequence $2 \leq i_1 < \cdots < i_l \leq n-1$ such that the set $p_{i_1 \cdots i_l}(X \cap (\xi_1 \times R^{n-1}))$ contains a nonempty open subset of $\xi_1 \times R_{i_1} \times \cdots \times R_{i_l}$ and a fortiori contains a ball $\xi_1 \times B_{i_1 \cdots i_l}(q, \varepsilon)$ with $(q, \varepsilon)$ rational. If $l = k-1$, this yields (2) and thus $\xi_1 \in S_{i_1 \cdots i_{k-1}}(q, \varepsilon)$. If $l = k$, we consider the projection

$$p : R_1 \times R_{i_1} \times \cdots \times R_{i_{k-1}} \times R_{i_k} \to R_1 \times R_{i_1} \times \cdots \times R_{i_{k-1}}$$

and note that

$$p(\xi_1 \times B_{i_1 \cdots i_l}(q, \varepsilon)) = \xi_1 \times B_{i_1 \cdots i_{k-1}}(p(q), \varepsilon)$$

and $(p(q), \varepsilon)$ is rational. Consequently, $p_{i_1 \cdots i_{k-1}}(X \cap (\xi_1 \times R^{n-1}))$ contains $\xi_1 \times B_{i_1 \cdots i_{k-1}}(p(q), \varepsilon)$ and therefore $\xi_1 \in S_{i_1 \cdots i_{k-1}}(p(q), \varepsilon)$. Formula (3) is thus established.

If a given set $S_{i_1 \cdots i_{k-1}}(q, \varepsilon)$ intersects a nondegenerate interval $I \subset R_1$ in a set $D$ which is dense in $I$, then by (2),

$$D \times B_{i_1 \cdots i_{k-1}}(q, \varepsilon) \subset p_{i_1 \cdots i_{k-1}}(X).$$

Since $p_{i_1 \cdots i_{k-1}}(X)$ is compact and $\bar{D} = I$, (4) implies

$$I \times B_{i_1 \cdots i_{k-1}}(q, \varepsilon) \subset p_{i_1 \cdots i_{k-1}}(X).$$

Consequently, in $R_1 \times R_{i_1} \times \cdots \times R_{i_{k-1}}$

$$\text{Int } p_{i_1 \cdots i_{k-1}}(X) \neq \varnothing.$$
We have thus either a projection
\[ p_{i_1\ldots i_{k-1}}: \mathbb{R}^n \to R_{i_1} \times \cdots \times R_{i_{k-1}}, \quad 1 \leq i_1 < \cdots < i_{k-1} \leq n, \]
satisfying (6), or every set \( S_{i_1\ldots i_{k-1}}(q, \varepsilon) \) is nowhere dense in \( R_1 \). However, in the latter case, by Baire's theorem, \( R_1 \cup S_{i_1\ldots i_{k-1}}(q, \varepsilon) \) must be dense in \( R_1 \). It then follows from (3) that \( S_1 \) too is a dense subset of \( R_1 \).

The same argument applies to any other \( j \in \{1, \ldots, n\} \) and we conclude that either there is a projection \( p_{i_1\ldots i_k}: \mathbb{R}^n \to R_{i_1} \times \cdots \times R_{i_k}, 1 \leq i_1 < \cdots < i_k \leq n \), such that \( j \in \{i_1, \ldots, i_k\} \) and \( \text{Int} p_{i_1\ldots i_k}(X) \neq \emptyset \) in \( R_{i_1} \times \cdots \times R_{i_k} \) or the set \( S_j \subset R_j \) of all \( f, e, \xi \in \mathbb{R} \) satisfying
\[ \dim(X \cap (R_1 \times \cdots \times R_{j-1} \times \xi_j \times R_{j+1} \times \cdots \times R_n)) \leq k - 2 \]
is dense in \( R_j \).

However, \( S_j \) cannot be dense in \( R_j \) for all \( j \in \{1, \ldots, n\} \). Indeed, that would imply that every point \( x \in X \) admits arbitrarily small neighborhoods \( U=(\alpha_1, \beta_1) \times \cdots \times (\alpha_n, \beta_n) \subset \mathbb{R}^n \), where \( \alpha_j, \beta_j \in S_j \) for all \( j \). Since, by (7), the boundary of \( U \) meets \( X \) in a set of dimension \( \leq k-2 \), we would have \( \dim X \leq k-1 \), which contradicts the assumption. This completes the proof of the theorem.

**Remark 1.** For \( k = n \) we have here an alternate proof for the fact that an \( n \)-dimensional compact subset \( X \subset \mathbb{R}^n \) has a nonempty interior.

**Remark 2.** A compact subset \( X \subset \mathbb{R}^n \) need not be of dimension \( \dim X \geq k \) if it admits a projection \( p_{i_1\ldots i_k}: \mathbb{R}^n \to R_{i_1} \times \cdots \times R_{i_k} \) with \( \dim p_{i_1\ldots i_k}(X) = k \). E.g., let \( I = [0, 1] \) and let \( f: I \to I^2 \) be a continuous surjection (\( I^2 \) is a Peano continuum). Then \( X = \{t \times f(t) | t \in I\} \subset \mathbb{R}^3 \) is an arc and \( \dim p_{i_1}(X) = 2 \).

**Remark 3.** The conclusion of the theorem remains true if one weakens the assumptions to \( X \) being a closed \( k \)-dimensional subset of \( \mathbb{R}^n \). Indeed, every closed \( X \) is the union of a sequence of compact subsets \( X_i \subset \mathbb{R}^n, i = 1, 2, \ldots \). Since \( k = \dim X = \max\{\dim X_i | i = 1, 2, \ldots\} \), there is an \( i \) such that \( \dim X_i = k \) and the conclusion follows from the one in the compact case.

**Reference**


**Department of Mathematics, University of Pittsburgh, Pittsburgh, Pennsylvania 15213**

*Current address*: Institute of Mathematics, University of Zagreb, Zagreb, Yugoslavia