FIELDS OF CONSTANTS OF INFINITE HIGHER DERIVATIONS

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Abstract. Let K be a field of characteristic \( p \neq 0 \), and let \( P \) be its maximal perfect subfield. Let \( h \) be a subfield of \( K \) containing \( P \) such that \( K \) is separable over \( h \). We prove: Every regular subfield of \( K \) containing \( h \) is the field of constants of a set of higher derivations on \( K \) if and only if (1) the transcendence degree of \( K \) over \( h \) is finite, and (2) \( K \) has a separating transcendency basis over \( h \). This result leads to a generalization of the Galois theory developed in [4].

I. Introduction. Let \( K \) be a field of characteristic \( p \neq 0 \), and let \( P \) be its maximal perfect subfield. If \( h \) is the field of constants of a set of higher derivations on \( K \), then \( h \) is a regular subfield of \( K \) containing \( P \). This paper is concerned with determining when every regular subfield of \( K \) containing \( h \) (and hence \( P \)) is the field of constants of a set of higher derivations on \( K \). Necessary and sufficient conditions are shown to be (1) the transcendence degree of \( K \) over \( h \) is finite, and (2) \( K \) has a separating transcendency basis over \( h \). This is Corollary (4.2). This result leads to an immediate extension of the Galois theory developed in [4]. In part, we can restate the main result of [4] as follows: Assume \( K \) has a finite separating transcendency basis over a subfield \( h \) containing \( P \). Then there exists a one-to-one correspondence between regular subfields of \( K \) containing \( h \) and Galois subgroups of \( H^\infty(K) \). (The characterization of Galois subgroups remains the same.) Moreover, (4.2) shows this to be the most general condition on \( K \) relative to \( h \) under which all regular subfields of \( K \) containing \( h \) will be fields of constants of groups of higher derivations on \( K \).

II. Definitions and preliminary results. Throughout this paper, \( K \) will be a field of characteristic \( p \neq 0 \). A higher derivation on \( K \) is a sequence \( d = \{d_i | 0 \leq i < \infty \} \) of additive maps of \( K \) into \( K \) such that

\[
d_r(ab) = \sum \{d_i(a)d_j(b) \mid i + j = r \}
\]

and \( d_0 \) is the identity map. The set \( H^\infty(K) \) of all higher derivations on \( K \)
is a group with respect to the composition $d \circ e = f$ where

$$f_i = \sum \{d_m e_n \mid m + n = i\}$$

[1, Theorem 1, p. 33]. Note that the first nonzero map (of subscript > 0) is a derivation. The field of constants of a subset $G \subseteq H^\alpha(K)$ is \{$a \in K \mid d_i(a) = 0, i > 0, (d_i) \in G$\}. $H^\alpha(K)$ will denote the group of all higher derivations on $K$ whose field of constants contains the subfield $h$.

(2.1) [2, Theorem 1]. Let $B$ be a $p$-basis for $K$ and let $f: A \times B \to K$ be an arbitrary function. There is a unique $(d_i) \in H^\alpha(K)$ such that for each $b \in B$ and $i \in \mathbb{Z}$, $d_i(b) = f(i, b)$.

(2.2) [3, p. 436]. Let $(d_i) \in H^\alpha(K)$ and $a \in K$. Then $d_{ip}(a^p) = (d_i(a))^p$ and if $p$ and $j$ are relatively prime, $d_{ij}(a^p) = 0$.

A field $K$ is a regular extension of a subfield $h$ if $K/h$ is separable and $h$ is algebraically closed in $K$.

(2.3) [4, Theorem 2.3]. Let $h$ be the field of constants of a set of higher derivations on $K$. Then $K$ is a regular extension of $h$.

(2.4) Lemma. The field of constants of $H^\alpha(K)$ is $P$, the maximal perfect subfield of $K$.

Proof. Let $a \in K \setminus P$. If \{a, $\alpha a^{-1}$, $\alpha^2 a^{-1}$, $\cdots$\} $\subseteq K$, then $P(a, \alpha a^{-1}, \alpha^2 a^{-1}, \cdots)$ would also be perfect, contrary to the assumption that $P$ is maximal. Thus there exists $n \geq 0$ such that $\alpha^p a^{-n} \in K \setminus P$. Let \{a$^p$\}$^\square \cup T$ be a $p$-basis for $K$, and define $d = \{d_i\}$ by $d_1(\alpha^p a^{-n}) = 1, d_i(t) = 0 \forall t \in T, d_i(x) = 0, x \in \{\alpha^p a^{-n}\} \cup T, 1 < i < \infty$. Then

$$d_{ip}(a) = d_{ip}((\alpha^p a^{-n})^p) = (d_i(\alpha^p a^{-n}))^p = 1$$

by (2.2). Thus the field of constants of $H^\alpha(K)$ is contained in $P$. Applying (2.2) shows $P$ is contained in the field of constants of $H^\alpha(K)$, and the lemma is established.

III. Higher derivations and separating transcendency bases. As before, $K$ is a field of characteristic $p \neq 0$ with maximal perfect subfield $P$. Throughout this section we assume the transcendence degree of $K/P$ (tr d(K/P)) is finite.

(3.1) Theorem. Let $K \supseteq h \supseteq P$ be fields and assume $K$ has a separating transcendency basis over $h$ and $h$ is algebraically closed in $K$. Then $h$ is the field of constants of a set of higher derivations on $K$.

Proof. Let \{x_1, \cdots, x_n\} be a separating transcendency basis, and hence a relative $p$-basis [5, Theorem 15, p. 384], for $K/h$, and let $T$ be a $p$-basis for $h$. Since $K/h$ is separable, \{x_1, \cdots, x_n\}$\cup T$ is a $p$-basis for $K$. Define $\mathcal{F} = \{d^1, \cdots, d^n\}$ by

$$d_i^1(x_i) = 1, \quad d_i^1(x) = 0, \quad x \in \{x_1, \cdots, x_1, \cdots, x_n\} \cup T, 1 \leq i \leq n,$$
and
\[ d_i^j(x) = 0, \quad x \in \{x_1, \cdots, x_n\} \cup T, \quad 1 \leq i \leq n, \quad 1 < j < \infty. \]

Since \( d_i^j(t) = 0 \) \( \forall t \in T, \quad 1 \leq i \leq n, \quad 1 \leq j < \infty, \) \( h \) is contained in the field of constants of \( \mathcal{F} \). By [4, Theorem 3.2], the transcendence degree of \( K \) over the field of constants of \( \mathcal{F} \) is \( n \), and hence \( h \) is the field of constants of \( \mathcal{F} \).

(3.2) Example. The condition of Theorem (3.1) is not necessary. Consider the following example [5, Example 10, p. 389]. Let \( P \) be a perfect field and \( Z = \{z_1, z_2, \cdots\} \) a denumerable set of elements algebraically independent over \( P \). Let \( P(Z^{\infty}) \) be the perfect field
\[ P(Z^{\infty}) = P(Z, Z^{-1}, Z^{-2}, \cdots). \]

Let \( y, u_0 \) be algebraically independent over \( P(Z^{\infty}) \), and define quantities \( u_n \) recursively by
\[ u_n = y^{p^n-1} + Z_n u_{n-1} \quad (n = 1, 2, \cdots). \]

Let \( K = P(Z^{\infty}, y, u_0, u_1^{p^{-1}}, u_2^{p^{-2}}, \cdots, u_n^{p^{-n}}, \cdots) \).

Mac Lane has shown \( P(Z^{\infty}) \) is the maximal perfect subfield of \( K \), \( K \) has \( \{y, u_0\} \) as a transcendency basis over \( P(Z^{\infty}) \), and \( \{y\} \) is a \( p \)-basis for \( K \). Thus \( K \) does not have a separating transcendency basis over \( P(Z^{\infty}) \), but by (2.4), \( P(Z^{\infty}) \) is the field of constants of \( H^{\infty}(K) \).

This example also shows that not every regular subfield \( h \) of \( K \) containing \( P \) is the field of constants of a set of higher derivations. Let \( h \) be the algebraic closure of \( P(Z^{\infty}, y) \) in \( K \). Since \( \{y\} \) is \( p \)-independent in \( K \), \( K/h \) is separable and hence regular. Since \( \text{tr} \text{d}(K/P(Z^{\infty})) = 2 \), \( h \neq K \). Since \( \{y\} \) is a \( p \)-basis for \( K \), the null set \( \emptyset \) is a relative \( p \)-basis for \( K/h \) and hence by (2.1) \( H^{\infty}_K(K) = \{0\} \) and \( h \) is not the field of constants of any set of higher derivations on \( K \).

Let \( h \) be a subfield of \( k \) containing \( P \) such that \( K \) is separable over \( h \) and assume \( \text{tr} \text{d}(K/h) < \infty \).

(3.3) Theorem. Every regular subfield \( k \) of \( K \) containing \( h \) is the field of constants of a set of higher derivations on \( K \) if and only if \( K \) has a separating transcendency basis over \( h \).

Proof. If \( K \) has a separating transcendency basis over \( h \), then \( K \) has one over any regular subfield \( k \) containing \( h \) [5, Theorem 18, p. 387], and hence every regular subfield \( k \) containing \( h \) is the field of constants of a set of higher derivations (3.1). Conversely, assume \( K \) does not have a separating transcendency basis over \( h \). Let \( T \) be any relative \( p \)-basis for \( K \) over \( h \). Since \( T \) is algebraically independent over \( h \), in view of [5, Theorem 13, p. 383], \( T \) cannot be a transcendency basis for \( K \) over \( h \).
Thus if we let \( k \) be the algebraic closure of \( h(T) \) in \( K \), \( k \not= K \), \( k \) is a regular subfield of \( K \) (since \( K/k \) preserves \( p \)-independence) and as in (3.2) \( H_k^\infty(K) = \{0\} \). Thus the theorem follows.

(3.4) Corollary. The following are equivalent.

(1) There exists a transcendency basis \( T \) for \( K \) over \( h \) such that \( K^{p^n} \) is a separable extension of \( h(T) \).

(2) Every regular subfield of \( K \) containing \( h \) is the field of constants of a set of higher derivations on \( K \).

(3) \( K \) has a separating transcendency basis over \( h \).

Proof. The equivalence of (1) and (3) is [5, Theorem 6, p. 375]. (3.3) shows (2) equivalent to (3).

IV. Transcendence degree of \( K/h = \infty \).

(4.1) Theorem. Let \( K \) be a field of characteristic \( p \neq 0 \). Let \( h \) be a subfield of \( K \) containing \( P \) such that \( K/p^n \) is a separable extension of \( h(T) \) and assume the transcendence degree of \( K \) over \( h \) is infinite. Then there exists a regular subfield \( k \) of \( K \) containing \( h \) which is not the field of constants of any set of \( p \)-bases on \( K \).

Proof. Let \( T \) be any relative \( p \)-basis for \( K \) over \( h \). If \( |T| < \infty \), let \( k \) be the algebraic closure of \( h(T) \) in \( K \). Then \( K \) is regular over \( k \) (\( K/k \) preserves \( p \)-independence) and since \( \varnothing \) is a relative \( p \)-basis for \( K \) over \( k \), \( H_k^\infty(K) = \{0\} \) and \( k \) is the desired subfield. If \( |T| = \infty \), let \( T = \{x_1, x_2, \ldots\} \cup S \). Let \( k_1 \) be the algebraic closure of \( h(s) \) in \( K \). Then \( \{x_1, x_2, \ldots\} \) is a relative \( p \)-basis for \( K \) over \( k_1 \). Elementary calculations show \( \{x_1 x_2^p, x_2 x_3^p, \ldots, x_n x_{n+1}^p, \ldots\} \) is also a relative \( p \)-basis. Since \( K/k_1 \) is separable, \( \{x_1 x_2^p, x_2 x_3^p, \ldots\} \) is algebraically independent over \( k_1 \). We claim \( \{x_1, x_1 x_2^p, x_2 x_3^p, \ldots\} \) is also algebraically independent over \( k_1 \). If not, \( \{x, x_1 x_2^p, \ldots, x_{n-1} x_n^p\} \) must be algebraically dependent over \( k_1 \) for some \( n \). But \( k_1(x_1, x_2, \ldots, x_n) \) is algebraic over \( k_1(x_1, x_1 x_2^p, \ldots, x_{n-1} x_n^p) \) and hence

\[
\text{tr} \, \text{d}(k_1(x_1, \ldots, x_n)/k_1) < n,
\]

a contradiction. Thus \( \{x_1, x_1 x_2^p, \ldots\} \) is algebraically independent over \( k_1 \).

Let \( k \) be the algebraic closure of \( k_1(x_1 x_2^p, x_2 x_3^p, \ldots) \) in \( K \). Then \( k \) is a regular subfield of \( K \), \( k \neq K \), and \( \varnothing \) is a relative \( p \)-basis for \( K \) over \( k \). Thus \( H_k^\infty(K) = \{0\} \) and \( k \) is not the field of constants of any set of higher derivations on \( K \).

(4.2) Corollary. Let \( K \) be a field of characteristic \( p \neq 0 \). Let \( h \) be a subfield of \( K \) containing \( P \) such that \( K \) is separable over \( h \). Then every regular subfield of \( K \) containing \( h \) is the field of constants of a set of higher derivations on \( K \).
derivations on $K$ if and only if (1) the transcendence degree of $K$ over $h$ is finite and (2) $K$ has a separating transcendency basis over $h$.

The Galois theory established in [4] required that $K$ be finitely generated over the distinguished regular subfields. In view of (4.2) we see that the correspondence can be extended to regular subfields $h$ such that $K$ has a finite separating transcendency basis over $h$. In part, the Galois correspondence can now be stated as follows.

(4.3) **Theorem.** Assume $K$ has a finite separating transcendency basis over a regular subfield $h$ containing $P$. Then there exists a one-to-one correspondence between the regular subfields of $K$ containing $h$ and Galois subgroups of $H_h^\infty(K)$.

The characterization of the Galois subgroups remains the same as in [4]. Moreover, (4.2) shows the condition that $K$ have a finite separating transcendency basis over $h$ to be the most general we can impose and maintain a complete correspondence in that all regular subfields of $K$ containing $h$ will be fields of constants of sets of higher derivations.

**References**


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