THE STRICT TOPOLOGY FOR P-SPACES

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Abstract. A P-space is a completely regular Hausdorff space X in which every $G_δ$ is open. It is shown that the generalized strict topologies $β$ and $β_0$ coincide on $C^*(X)$, and that strong measure-theoretic properties hold; in particular, $(C^*(X), β)$ is always a strong Mackey space. As an application, an example is constructed of a non-quasi-complete locally convex space in which closed totally bounded sets are compact.

1. Introduction. All topological spaces considered here are assumed to be completely regular Hausdorff. If X is such a space, then $C^*(X)$ denotes the space of bounded real-valued continuous functions on X. The strict topology $β$ on $C^*(X)$ was introduced by Buck [6] for locally compact X; combining his ideas with the measure-theoretic concepts of Varadarajan [18], Sentilles [17] considered locally convex topologies $β_0$, $β$, and $β_1$ on $C^*(X)$, X completely regular, which yield the spaces $M_t(X)$, $M_r(X)$, and $M_σ(X)$ of tight, $τ$-additive, and $σ$-additive Baire measures as duals. A useful reference for the notions of uniform $σ$- and $τ$-additivity is [10]. We assume the results of these papers as needed.

The principal source of information about P-spaces is the Gillman-Jerison text [11]. Nondiscrete P-spaces are remote from the locally compact spaces studied in measure theory; indeed every compact subset of such a space is finite. Nevertheless, it is shown here that certain aspects of the strict topology in the locally compact case remain valid for P-spaces. For example, the topologies $β$ and $β_0$ coincide, and $(C^*(X), β)$ is always a strong Mackey space. The latter result can be obtained using either the well-known techniques of Conway [8] for paracompact locally compact spaces or via the more recent ideas of uniform $σ$- and $τ$-additivity, giving an opportunity to compare and contrast these methods. Every discrete space is a P-space, and some results of this paper can be viewed as extensions of the work of Collins [7] in the discrete case. On the other
hand, the existence of a P-space which admits no complete uniform structure \([11, 9L]\) permits the construction of counterexamples to two recent conjectures in functional analysis and topological measure theory.

The following properties of P-spaces will be needed: every zero-set of a real-valued continuous function is open (indeed the Baire sets of a P-space are precisely the clopen subsets). If \((f_n)\) is a uniformly bounded sequence of functions in \(C^*(X)\), then \(f(x) = \sup f_n(x)\), and \(g(x) = \inf f_n(x)\) are also members of \(C^*(X)\). The latter property resembles (but is stronger than) the requirement that \(C^*(X)\) with the usual ordering be a conditionally \(\sigma\)-complete lattice.

2. Properties of \(\beta\) and \(\beta_0\) for P-spaces. For any space \(X\), let \(X_d\) denote the underlying set of \(X\), endowed with the discrete topology. Then \(C^*(X)\) can be considered as a subspace of \(C^*(X_d)\).

**Theorem 2.1.** If \(X\) is a P-space, then (a) \((C^*(X), \beta_0)\) is topologically isomorphic under the inclusion map to a subspace of \((C^*(X_d), \beta_0)\), and the latter space is its completion; (b) \((C^*(X), \beta_0)\) is sequentially complete; (c) \(M_t(X) = M_t(X_d) = \mathbb{P}(X)\).

**Proof.** Since every compact subset of \(X\) is finite, the associated \(k\)-space of \(X\) is \(X_d\); now (a) follows from Theorem 6 of \([10]\). The result of (b) is an easy consequence of the fact that, in a P-space, the pointwise limit of any uniformly bounded sequence of continuous functions is again continuous. The final result was proved by Babiker \([2]\).

Now we use the Conway-LeCam technique to show that \(\beta\) and \(\beta_0\) are identical in our setting.

**Theorem 2.2.** If \(X\) is a P-space, then \((C^*(X), \beta_0)\) is a strong Mackey space, and \(\beta = \beta_0\).

**Proof.** Since \(\beta\) and \(\beta_0\) yield the same dual space (2.1(c)), and always \(\beta_0 \leq \beta\), the first result implies the second. Let \(A\) be a subset of \(M_t(X)\) such that every sequence in \(A\) has a weak*-cluster point (in \(M_t(X)\)). If \(A\) is not uniformly tight, then we can find \(\varepsilon_0 > 0\), pairwise disjoint compact (equivalently, finite) subsets \((D_n)\) of \(X\) and members \((\mu_n)\) of \(A\) with

\[
|\mu_n| \left( X \setminus \bigcup_{i=1}^{n-1} D_i \right) > \varepsilon_0
\]

and

\[
|\mu_n| \left( X \setminus \bigcup_{i=1}^n D_i \right) < \varepsilon_0/4 \forall n.
\]

Since every \(G_\delta\) in \(X\) is open, every pairwise disjoint sequence of closed sets in \(X\) is discrete. Applying this to the sequence of compact sets \((D_n)\), we can easily obtain a sequence \((F_n)\) of pairwise disjoint closed sets with each
$D_n$ contained in the interior of $F_n$. Then $(F_n)$ is also a discrete sequence of sets. Let $D_n = \{x_{i_n} : 1 \leq i \leq i_n\}$, and for each $n$, choose $f_n \in C^*(X)$ with $f_n(x_{i_n}) = \text{sgn} \mu_n((x_{i_n}), f_n|X\setminus F_n \equiv 0$, and $\|f_n\| \leq 1$. Then the map $T : l^\infty \to C^*(X)$ defined by $T(\alpha) = \sum_{n=1}^{\infty} \alpha_n f_n$ is $\beta_0 - \beta_0$ continuous. Thus the adjoint $T^*$ maps $M_1(X)$ into $l^1$, and $T^*A$ is relatively $\sigma(l^1, l^\infty)$-countably compact, hence relatively norm compact. Hence there is a positive integer $n_0$ such that $|T^*(\mu_n(e_n))| < \varepsilon_0 / 2 \forall n \geq n_0$, where $e_n$ is the $n$th unit vector. However, it is easily verified that $|T^*(\mu_n(f_n))| > \varepsilon_0 / 2$, and we have a contradiction.

We now give a characterization of $P$-spaces and discrete spaces $X$ in terms of behavior of $B = \{f \in C^*(X) : \|f\| \leq 1\}$.

**Theorem 2.3.** Let $X$ be completely regular Hausdorff. Then

(a) $B$ is $\beta$- (or $\beta_0$-) compact if and only if $X$ is discrete;
(b) $B$ is $\beta$- (or $\beta_0$-) countably compact if and only if $X$ is a $P$-space;
(c) $B$ is $\beta_0$-totally bounded if and only if every compact subset of $X$ is finite.

**Proof.** (a) If $B$ is $\beta_0$-compact and $p \in X$, let $(f_a)$ be a net in $B$ which converges pointwise to the characteristic function of $\{p\}$. Then any $\beta_0$-cluster point of $(f_a)$ necessarily coincides with this characteristic function, so $\{p\}$ is open and $X$ is discrete. Conversely if $X$ is discrete, then $B$ is $\beta$-totally bounded [7, Theorem 4.1] and $\beta$-complete, hence $\beta$-compact.

(b) If $B$ is $\beta_0$-countably compact, then, replacing $\{p\}$ and $(f_a)$ in (a) by a zero-set $Z$ and sequence $(f_n)$, we find $X$ to be a $P$-space (every zero-set is open). Conversely, if $X$ is a $P$-space and $(f_n)$ is a sequence in $B$, define an equivalence relation on $X$ by $x \sim y$ iff $f_n(x) = f_n(y) \forall n$. Each equivalence class is a zero-set, hence open, so the quotient space $Y$ is discrete. Let $\pi : X \to Y$ be the quotient map; there is a sequence $(g_n)$ in the unit ball of $C^*(Y)$ with $g_n \circ \pi = f_n \forall n$. From (a), $(g_n)$ has a $\beta_0$-cluster point $g_0$, and $g_0 \circ \pi$ is a $\beta_0$-cluster point of $(f_n)$. Since $\beta_0 = \beta$ for $X$ a $P$-space, the result follows.

(c) Since $\beta_0$ and the compact-open topology agree on $B$, this result is immediate.

Now we have that $X$ a $P$-space $\Rightarrow B$ is $\beta$-totally bounded $\Rightarrow$ compact subsets of $X$ are finite. However, neither implication can be reversed as we now show. Varadarajan [18, pp. 225–227] discusses two spaces which may be described as follows: Let $N$ denote the set of positive integers, and let $\mathcal{F}$ be the filter of subsets of $N$ which have density one [11, 6U], with $p$ a fixed cluster point of $\mathcal{F}$ in $\beta N$. Let $\mathcal{U}$ be the unique ultrafilter on $N$ which refines $\mathcal{F}$ and converges to $p$. Then $\mathcal{N} \cup \{p\}$ can be topologized by requiring each point of $\mathcal{N}$ to be open and neighborhoods of $p$ to be of the form $\{p\} \cup F$, where $F \in \mathcal{F}$ or $F \in \mathcal{U}$; call the resulting spaces $V$ and $E$ ($E$ has the relative topology of $\beta N$). It can be shown
(using, for example, Theorem 2.13 of [15]) that $\beta=\beta_0$ on $C^*(E)$. Since compact subsets of $E$ are finite, $B \subset C^*(E)$ is then $\beta$-totally bounded, but $E$ is not a $P$-space.

On the other hand, compact subsets of $V$ are finite, and $\beta$ and $\beta_0$ yield the same dual space for $C^*(V)$ [15, Proposition 3.4], yet $B \subset C^*(V)$ is not $\beta$-totally bounded. If it were, then, by a duality result of Grothendieck [14, p. 266], each $\beta$-equicontinuous subset of $M_\tau(V)$ would be norm-totally bounded. For each $x \in V$, let $\delta(x)$ be the point mass at $x$; define $\mu_n=n^{-1}(\sum_{i=1}^{n} \delta(i))$, $\mu_0=\delta(p)$. Then $(\mu_n)$ is weak*-convergent to $\mu_0$ in $M_\tau^+(V)$ [18, p. 226] and so $A=\{\mu_n: n \geq 0\}$ is $\beta$-equicontinuous [17, Theorem 5.2], but, as is easily seen, not norm-totally bounded.

3. Measure-theoretic properties of $P$-spaces. Any space $X$ for which $M_\sigma(X)=M_\tau(X)$ is realcompact; Babiker [2], [3] has shown that the converse is true for $P$-spaces (with certain cardinality assumptions). By minor modifications of his arguments it can be shown that (in the terminology of [19]) a $P$-space satisfies $M_\tau=M_\sigma$ if and only if it is topologically complete (no cardinality assumptions needed). However, a topologically complete $P$-space need not be paracompact, or even normal [1].

The next result shows that $P$-spaces have the following curious property: any set of $\tau$-additive measures which is well-behaved with respect to sequences (uniformly $\sigma$-additive) is necessarily well-behaved with respect to nets (uniformly $\tau$-additive). This is true in spite of the fact that $M_\sigma(X) \neq M_\tau(X)$ for certain $P$-spaces $X$ (§4). In the following, let $\xi$ denote the family of uniformly bounded equicontinuous subsets of $C^*(X)$, and let $\mathcal{T}(\xi)$ denote the topology on $M_\tau(X)$ of uniform convergence on members of $\xi$. Since sequences in $C^*(X)$ which are either norm convergent to 0 or monotone decreasing and pointwise convergent to 0 are equicontinuous, it is easy to see that $(M_\sigma(X), \mathcal{T}(\xi))$ is a complete locally convex space.

Theorem 3.1. If $X$ is a $P$-space, then the following conditions on a subset $H$ of $M_\tau(X)$ are equivalent:

(a) uniformly $\tau$-additive;
(b) relatively weak*-compact in $M_\tau(X)$;
(c) every sequence in $H$ has a weak*-cluster point in $M_\tau(X)$;
(d) every sequence in $H$ has a weak*-cluster point in $M_\sigma(X)$;
(e) uniformly $\sigma$-additive;
(f) norm-totally bounded;
(g) $\mathcal{T}(\xi)$-totally bounded.

Proof. The implications (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d)$\Leftarrow$(e) and (f)$\Rightarrow$(g)$\Rightarrow$(d) are true for any $X$ ((d)$\Leftarrow$(e) was shown by Varadarajan [18, p. 203], and...
(g)⇒(d) follows from \( T(\xi) \)-completeness of \( M_\sigma(X) \). For \( X \) a \( P \)-space, we show (e)⇒(a)⇒(f). If \( H \) is uniformly \( \sigma \)-additive, then so are \( H^+=\{\mu^+:\mu\in H\} \) and \( H^-\{\mu^-:\mu\in H\} \) [10, p. 120]; hence we may as well assume that \( H \) consists of nonnegative measures. If \( H \) is not uniformly \( \tau \)-additive, there is a net \((f_\alpha)\) in \( C^*(X) \) and \( \varepsilon_0>0 \) such that
\[
\sup\{\mu(f_\alpha):\mu\in H\} > \varepsilon_0 \forall \alpha.
\]
Using the \( \tau \)-additivity of each member of \( H \), we can find a sequence \( \alpha_1>\alpha_2>\cdots \) of indices and members (\( \mu_n \)) of \( H \) such that \( \mu_n(f_{\alpha_{n+1}})<\varepsilon_0/2^n \). Let \( f_0=\inf f_{\alpha_n} \). Then \( f_0 \in C^*(X) \) and \( (f_{\alpha_n}-f_0)\downarrow 0 \), so \( \mu(f_{\alpha_n}-f_0)\to 0 \) uniformly with respect to \( \mu \in A \). This implies the existence of an integer \( n_0 \) such that \( \mu_{\alpha_n}(f_{\alpha_{n+1}})<\varepsilon_0/4 \) for \( n \geq n_0 \), a contradiction.

Since the uniformly \( \tau \)-additive sets are precisely the \( \beta \)-equicontinuous sets, (a)⇒(f) follows from 2.3 and the Grothendieck result mentioned in the previous section.

Note that the equivalence of (a) and (c) furnishes an alternate proof that \((C^*(X), \beta)\) is a strong Mackey space, independent of 2.2.

**Corollary 3.2.** If \( X \) is a \( P \)-space, then (a) the norm and weak* topologies agree on \( \beta \)-equicontinuous subsets of \( M_\sigma(X) \); (b) \( M_\sigma(X) \) is weak*-sequentially complete; in fact, every weak*-Cauchy sequence in \( M_\sigma(X) \) is norm-convergent.

**Proof.** Since \( M_\sigma(X) \) is a norm-closed subspace of \( M(X) \), the Banach dual of \( C^*(X) \), the first assertion follows from the equivalence of (a) and (f) in 3.1. It is known [18, p. 195] that \( M_\sigma(X) \) is weak*-sequentially complete. Thus a weak*-Cauchy sequence in \( M_\sigma(X) \) satisfies 3.1(d), so it is \( \beta \)-equicontinuous, and the result follows.

As an immediate consequence of 3.2(a), the finest topology on the dual of \((C^*(X), \beta)\) which agrees with the weak* topology on \( \beta \)-equicontinuous sets is the norm topology. The corresponding result for discrete \( X \) was proved by Collins [7, Theorem 4.1]. Note that any weak*-convergent sequence in \( M(X) \) is weakly convergent; since \( \beta X \) is an \( F \)-space when \( X \) is a \( P \)-space, this follows from a result of Seever [16].

**4. A counterexample.** An example of a non-realcompact \( P \)-space \( S \) is recorded in [11, 9L]. Applying Shirota's theorem [11, p. 229] and 2.2, we have: \((C^*(S), \beta)\) is a strong Mackey space, although \( S \) admits no complete uniform structure. This resolves negatively a conjecture advanced by the author in [19]. The conjecture, however, remains open for the category of \( k \)-spaces (in particular, for locally compact spaces).

The class of locally convex spaces in which closed totally bounded sets are compact has been examined in [4] and [9]. Answering a question
posed by Buchwalter [5], Haydon [13] has given an example of a non-
quasi-complete space with this property. The example offered here,
obtained independently by the author, is of a very different sort.

Example 4.1. \((M(S), \mathcal{T}(\xi))\) is a non-quasi-complete locally convex
space in which closed totally bounded sets are compact.

Note that \(M(S)\) is identical as a vector space to \(l^1(S_d)\), where \(S_d\) has
cardinal \(\aleph_2\). The fact that closed totally bounded sets are compact follows
readily from 3.1. On the other hand, there is a natural embedding \(j: S \to (M(S), \mathcal{T}(\xi))\). Arguing as in [19], it can be shown that \(j(S)\) is \(\mathcal{T}(\xi)\)-
closed and bounded in \(M(S)\), yet the \(\mathcal{T}(\xi)\)-closure of \(j(S)\) in \(M_d(S)\) is
(a copy of) the Hewitt real-compactification of \(S\), hence properly contains
\(j(S)\). Thus \(j(S)\) is not \(\mathcal{T}(\xi)\)-complete.

5. Possible extensions. The results obtained here for \(P\)-spaces are not
valid, in general, for other classes of highly disconnected spaces. For
example, assume the continuum hypothesis, and let \(p\) be a \(P\)-point of
\(\beta N\setminus N\) [11, 6V]. Then \(X = \beta N\setminus\{p\}\) is extremally disconnected and locally
compact. However, \((C^*(X), \beta)\) is not a Mackey space; the argument is
similar to that of Conway for the ordinals less than \(\omega_1\) [8, p. 481]. But
under the additional assumption that compact subsets of \(X\) are finite,
strong results for highly disconnected spaces have been obtained recently
by Haydon [12].

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