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INJECTIVE OBJECTS IN THE CATEGORY OF p-RINGS

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ABSTRACT. A p-ring (or generalized Boolean ring) P is a ring of fixed prime characteristic p in which $a^p=a$ for all a in P. In this paper P is partially ordered by a relation which is a generalization of the usual Boolean order. A subset S of P is then called quasi-orthogonal if $ab(a-b)=0$ for all a, b in S. It is shown that P is injective in the category of p-rings if and only if every quasi-orthogonal subset has a supremum under this partial order.

Sikorski [4] has shown that in the category $\mathscr{B}$ of Boolean rings the injective objects are the complete Boolean rings. The purpose of this paper is to present a generalization of this result to the category $\mathscr{P}$ of p-rings, where $\mathscr{P}$ is understood to be the category with objects rings P of fixed prime characteristic p in which $a^p=a$ for all $a\in P$ and with morphisms the usual ring homomorphisms.

If P is a p-ring, then the set $B(P)$ of idempotents of P is a Boolean ring under the multiplication of P and the new addition defined by $a\oplus b=a+b-2ab$. Batbedat [2] has used B to establish an isomorphism between $\mathscr{P}$ and $\mathscr{B}$. (See also Stringall [6].) Hence, the injective objects in $\mathscr{P}$ are simply those P for which $B(P)$ is a complete Boolean ring. In this paper the injective objects in $\mathscr{P}$ will be characterized by a kind of completeness of a particular partial order that is an extension of the usual partial order on the Boolean ring of idempotents.

DEFINITION 1. For $a, b\in P$, $a\leqq b$ if and only if $a^{p-1}b=a$.

It is easily shown that $\leqq$ is a partial order on P (see, e.g., Abian [1]), but in general $(P, \leqq)$ is not a lattice. It is, however, a lower semilattice with meet defined by $a\wedge b=a-a(a-b)^{p-1}$. A simple calculation shows that $a(b\wedge c)=ab\wedge ac$ and $a\wedge 0=0$ for $a, b, c\in P$.

Foster [3] has shown that every element a of P is uniquely expressible as a sum of elements $\{e_j(a):j\in Z_p\}$ of $B(P)$. In particular,

$$a=\sum_{j=1}^{p-1} je_j(a) \tag{1}$$

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where

$$(2) \qquad e_j(a) = (a - a(a - ja^{p-1})^{p-1})^{p-1}$$

and

$$(3) \qquad e_j(a)e_k(a) = 0 \quad \text{if } j \not\equiv k \pmod{p}.$$

The following lemma provides an alternative representation of a as the sum of meets of elements of P.

LEMMA 1. *If $a \in P$ and $j \in Z_p$, then $je_j(a)=a\wedge ja^{p-1}$.*

PROOF. First observe that if $b\leqq a$, then for any integer $n\geqq 1$, $a^n=((a-b)+b)^n=(a-b)^n+A+b^n$, where A has a factor of $b(a-b)=b(a-b^{p-1}a)=0$. Hence $a^n-b^n=(a-b)^n$. Letting $b=a(a-ja^{p-1})^{p-1}$ and $n=p-1$, it follows that

$$je_j(a) = j(a^{p-1} - a^{p-1}(a - ja^{p-1})^{p-1}) = a \wedge ja^{p-1}.$$

In general, not every pair of elements of P has a join. It is easily shown, however, that a and b have a join in P if $ab(a-b)=0$. In particular, $a\vee b=a+b-a^{p-1}b$. (Note that $ab(a-b)=0$ is equivalent to $a^{p-1}b=ab^{p-1}$.) More generally, if the set Q has an upper bound, then $ab(a-b)=0$ for all $a, b \in Q$.

DEFINITION 2. Elements a, b in P are said to be quasi-orthogonal if $ab(a-b)=0$. A subset Q of P is called quasi-orthogonal if its elements are pairwise quasi-orthogonal. P is then called quasi-orthogonally complete if every quasi-orthogonal subset has a supremum in P.

We now show that quasi-orthogonal elements have an important property in terms of the idempotents given in (2).

LEMMA 2. *If $ab(a-b)=0$ then $e_j(a)e_k(b)=0$ for $j\not\equiv k \pmod{p}$.*

PROOF. We may assume $j, k\not\equiv 0 \pmod{p}$. Then

$$\begin{aligned} je_j(a)ke_k(b) &= (a \wedge ja^{p-1})(b \wedge kb^{p-1}) \\ &= ab \wedge ja^{p-1}b \wedge ka^{p-1}b \wedge jka^{p-1}b^{p-1}. \end{aligned}$$

But for any x in P, $kx\wedge jx=kx-kx(kx-jx)^{p-1}=kx-kx(k-j)^{p-1}x^{p-1}=0$, and letting $x=ab^{p-1}$, we obtain $je_j(a)ke_k(b)=0$ and hence $e_j(a)e_k(b)=0$.

The next lemma makes use of the well-known fact (see, e.g., Sikorski [5, p. 60]) that if S is a subset of a Boolean ring R with supremum in R and if $m \in R$, then

$$(4) \qquad m(\sup S) = \sup(mS).$$

LEMMA 3. *If Q is a quasi-orthogonal subset of P such that* $\sup\{e_j(a): a \in Q\}$ *is in P, then*

$$\sup\{b^{p-1}e_j(a): a \in Q\} = e_j(b) \quad \text{for all } b \in Q, j \in Z_p.$$

PROOF. Note first that

$$b^{p-1}e_j(a) = (be_j(a))^{p-1} = \left(\sum_{j=1}^{p-1} ke_k(b)e_j(a)\right)^{p-1}$$
$$= (je_j(b)e_j(a))^{p-1} = e_j(b)e_j(a),$$

using (1) and Lemma 2. Hence

$$e_j(b) = e_j(b)\sup\{e_j(a): a \in Q\}$$
$$= \sup\{e_j(b)e_j(a): a \in Q\} = \sup\{b^{p-1}e_j(a): a \in Q\}.$$

We are now ready to prove an important theorem relating the completeness of $B(P)$ to the quasi-orthogonal completeness of P. For brevity, sums are assumed to be taken over $j, k = 1, \cdots, p-1$.

THEOREM. *If P is a p-ring and $B(P)$ the Boolean ring of idempotents of P, then P is quasi-orthogonally complete if and only if $B(P)$ is complete.*

PROOF. We first observe that since any subset S of $B(P)$ is quasi-orthogonal, if P is quasi-orthogonally complete, then $\sup S \in P$. However $(\sup S)^2 = \sup S$ from (4), so that $\sup S \in B(P)$ and $B(P)$ is complete.

Conversely, let Q be a quasi-orthogonal subset of P and $s = \sum j \sup\{e_j(a): a \in Q\}$. Because $e_j(a) \in B(P)$ for $a \in P$ and $j \in Z_p$, the various suprema are in P and hence s is in P.

Now choose $s \in Q$. Then $x^{p-1} \in B(P)$ and so

$$x^{p-1}s = \sum jx^{p-1}\sup\{e_j(a): a \in Q\}$$
$$= \sum j \sup\{x^{p-1}e_j(a): a \in Q\} = \sum je_j(x) = x,$$

using (4), Lemma 3, and (1). Thus $x \leqq s$ and x is an upper bound of Q.

On the other hand, assume u is any upper bound of Q. It follows from (4) and Lemma 2 that

$$\sup\{e_i(a): a \in Q\} \cdot \sup\{e_j(a): a \in Q\} = 0 \quad \text{if } i \not\equiv j \pmod p.$$

Hence,

$$s^{p-1}u = \sum j^{p-1}u(\sup\{e_j(a): a \in Q\}) = \sum \left(\sum k \sup\{e_k(u)e_j(a): a \in Q\}\right)$$

from (1) and (4). But if $a \leqq u$, then $a^{p-1}u = au^{p-1}$ and so from Lemma 2, $e_k(u)e_j(a) = 0$ if $k \not\equiv j \pmod p$. Also, a straightforward calculation shows that $e_j(u)e_j(a) = e_j(a)$, so $s^{p-1}u = \sum j \sup\{e_j(a): a \in Q\} = s$. Consequently, $s \leqq u$ and so s is the supremum of Q in P.

In light of the above theorem and the remarks in the introductory paragraphs, we obtain the following corollary.

COROLLARY. *A p-ring is an injective object in the category of p-rings if and only if it is quasi-orthogonally complete.*

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