PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 42, Number 1, January 1974

WEIGHTED REPRESENTATIONS OF A PRIMITIVE ALGEBRA

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ABSTRACT. Let L be a diagonable subspace of an associative algebra A with identity over a field F; that is, L is spanned by a set of pairwise commuting elements, and the linear transformations $\operatorname{ad} x: a \mapsto ax - xa$ for $x \in L$ are simultaneously diagonalizable. Denote the centralizer of L in A by $\mathscr{C}$. A module V over A or $\mathscr{C}$ is L-weighted if for some nonzero $v \in V$ and map $\lambda: L \to F$, $v(x-\lambda(x)1)^{n(x)}=0$ for each $x \in L$, and x-weighted if for some nonzero $v \in V$, $\lambda \in F$ and positive integer n, $v(x-\lambda 1)^n=0$. In this paper we give conditions under which the following statements are equivalent:

1. All irreducible modules over A and $\mathscr{C}$ are L-weighted.
2. For each $x \in L$, some irreducible A-module is x-weighted and x is algebraic over F.

Let A be an associative algebra with 1 over a field F of characteristic 0 which possesses a *diagonable subspace*. By this, we mean a subspace L spanned by a set of pairwise commuting elements such that $\{\operatorname{ad} x : x \in L\}$ is a simultaneously diagonalizable set of linear transformations of A, where for any x, $a \in A$, $\operatorname{ad} x: a \mapsto (a, x) \equiv ax - xa$. Corresponding to L, there is a collection Δ of maps $\alpha: L \to F$, called *roots* of L in A, such that A decomposes into $\bigoplus \sum_{\alpha \in \Delta} A_\alpha$, where $A_\alpha = A_\alpha(L) = \{a \in A : (a, x) = \alpha(x)a$ for every $x \in L\}$. A module V over any subalgebra of A containing L is said to be *L-weighted* if there is a map $\lambda: L \to F$, called a *weight* of L in V, and a nonzero $v \in V$, such that $v(x-\lambda(x)1)^n=0$ for $n=n(x) \in N$, the natural numbers, and each $x \in L$. If for some $x \in L$, $n \in N$, and $\lambda \in F$, $v(x-\lambda 1)^n=0$, then V is *x-weighted*. In fact, any weight or root is always a linear functional on L [**1**]. It is known [**1**] that should V be both L-weighted and irreducible, then V has a *weight space decomposition*, $V = \bigoplus \sum_{\lambda \in \Lambda} V_\lambda$ relative to the complete set Λ of weights of L in V. Here $V_\lambda = \{v \in V : v(x-\lambda(x)1)^{n(x)}=0$ for each $x \in L\}$ is the *weight space* corresponding to λ.

Received by the editors April 11, 1973.

AMS (MOS) subject classifications (1970). Primary 16A12, 16A20; Secondary 17B10.

Key words and phrases. Diagonable subspace, weighted representation.

[1] Part of this work appears in the author's doctoral dissertation written at the University of British Columbia under the supervision of Dr. C. T. Anderson.

Now $A_0(L)$ is a subalgebra of A containing 1 and L, for it is just the centralizer $\mathscr{C}$ of L in A. This centralizer plays an important role in the representation theory for A: we have shown elsewhere that a theorem of Lemire [2] concerning the weighted irreducible representations of a finite-dimensional Lie algebra over an algebraically closed field of characteristic 0 is a consequence of the fact that any Cartan subalgebra of such a Lie algebra is a diagonable subspace of the universal enveloping algebra. For a fixed map $\lambda: L \to F$, the irreducible L-weighted representations of A which have λ as a weight are in one-to-one correspondence with the same class of representations of $\mathscr{C}$, up to equivalence [1].

Now the mere existence of a diagonable subspace in A does not guarantee that all irreducible A-modules are weighted, as an example in [3] shows; furthermore, it is not clear that the existence of weight space decompositions for all irreducible A-modules implies such decompositions for all irreducible $\mathscr{C}$-modules. In this paper, we give a necessary and sufficient condition for all the representations of A and $\mathscr{C}$ to be L-weighted. The notation and conventions of our first paragraph will be everywhere employed.

THEOREM. *Let $A = \bigoplus \sum_{\alpha \in \Delta} A_\alpha$ be the decomposition of a primitive algebra A corresponding to a diagonable subspace L with centralizer $\mathscr{C}$. Suppose Δ is finite and L is finite dimensional. Then the following are equivalent*:

(1) *All irreducible modules of A and $\mathscr{C}$ are L-weighted.*

(2) *For each $x \in L$, x is algebraic and some irreducible A-module is x-weighted.*

EXAMPLE. We begin with an example intended to show the complexity of the problem of determining which representations of an algebra are weighted. Let A be the universal enveloping algebra of the two-dimensional nonabelian Lie algebra over a field F (of characteristic 0). Then A is just the ring of noncommuting polynomials over F in two indeterminates x and y, with $yx=(x+1)y$. One can verify that, in fact, $y^i f(x) = f(x+i1)y^i$ for any $i \in N$ and $f(x) \in F[x]$. The subspace Fx is diagonable and $A = \bigoplus \sum_{\alpha_n \in \Delta} A_{\alpha_n}$ where $\alpha_n : x \mapsto nx$ for any integer $n \geqq 0$. In fact, A_{α_n} is merely $F[x]y^n$. We prove first that A is primitive. To this end, note that $y+1$ is not invertible in A and so must be contained in a maximal right ideal J which has the property

$$(1) \qquad f(x)y^k \in J \quad \text{for } k > 0, \quad \text{and} \quad f(x) \in F[x] \text{ implies } f \equiv 0.$$

This follows by induction on n=degree f and k. For $n=0$, if αy^k and $y+1$ are both in J, then J contains 1 because these are relatively prime polynomials. Assuming the validity of (1) for polynomials f with degree less than n, suppose $f(x) \in F[x]$ has degree n and $f(x)y^k$ and $y+1$ are both in J. Then J must contain

$$(f(x)y^k, y + 1) = f(x)y^{k+1} - yf(x)y^k = (f(x) - f(x + 1))y^{k+1}.$$

Since $f(x)-f(x+1)$ has degree less than n, $f(x)=f(x+1)$ by induction. This means $\xi+m$ is a root of f for every integer $m \geqq 0$ and any root ξ of f. But this is impossible unless f has degree 0, and this possibility has already been eliminated. We therefore have (1). It follows immediately that J can contain no two-sided ideal of A except (0) because any such ideal T can be written $T=\bigoplus\sum_{\alpha_n\in\Delta} T\cap A_{\alpha_n}$ (§4 of [1]). Hence A/J is a simple, faithful A-module and A is primitive.

Now A/J is not weighted, for we can prove

$$\left(\sum_{i=0}^{n} a_i(x)y^i\right)(x - \alpha 1)^m \in J \quad \text{implies} \quad \sum_{i=0}^{n} a_i(x)y^i \in J \tag{2}$$

where $a_i(x) \in F[x]$, $i=0, \cdots, n$. For $n=0$, $a_0(x)(x-\alpha 1)^m \in J$ implies $a_0(x)=0$ or $x=\alpha 1$ by (1), and the latter possibility is false. If J contains

$$\left(\sum_{i=0}^{n} a_i(x)y^i\right)(x - \alpha 1)^m = \sum_{i=0}^{n} y^i a_i(x - i1)(x - \alpha 1)^m$$

then it also contains

$$(y + 1)^n b(x)(x - \alpha 1)^m = \sum_{i=0}^{n} \binom{n}{i} y^i b(x)(x - \alpha 1)^m, \qquad b(x) = a_n(x - n1)$$

because $y+1 \in J$, and so J contains the difference of these two elements; namely,

$$\sum_{i=0}^{n-1} y^i\left(a_i(x - i1) - \binom{n}{i} b(x)\right)(x - \alpha 1)^m$$
$$= \sum_{i=0}^{n-1} \left(a_i(x) - \binom{n}{i} b(x + i1)\right) y^i (x - \alpha 1)^m.$$

By induction, we may assume $a_i(x)=\binom{n}{i}b(x+i1)$; i.e. that $a_i(x)=\binom{n}{i}a_0(x+i1)$, for $i=0, \cdots, n-1$. Thus

$$\sum_{i=0}^{n} a_i(x)y^i = \sum_{i=0}^{n} \binom{n}{i} a_0(x + i1)y^i = \sum_{i=0}^{n} \binom{n}{i} y^i a_0(x) = (1 + y)^n a_0(x)$$

is in J. Hence we have (2) and A/J cannot be weighted.

On the other hand, some A-modules are weighted; for example, there is a maximal right ideal I of A containing x (because x is not invertible) and so A/I is an irreducible Fx-weighted A-module $((I+1)\alpha x=0$ for any $\alpha \in F)$.

PROPOSITION. *If L is a finite-dimensional diagonable subspace of A and V is an irreducible A-module, then V is L-weighted if and only if V is x-weighted for every $x \in L$.*

PROOF. If V is L-weighted, then V is x-weighted for every $x \in L$ by definition. Conversely, suppose $x_1, \cdots, x_k$ is a basis for L. By induction, we assume that there exist scalars $\lambda_1, \cdots, \lambda_{k-1}$, and positive integers $n_1, \cdots, n_{k-1}$, such that $v(x_i-\lambda_i 1)^{n_i}=0$ for some nonzero $v \in V$, $i=1, \cdots, k-1$. Now, because V is x_k-weighted, it is easy to see that V has a weight space decomposition relative to the diagonable subspace Fx_k, $V=\bigoplus\sum_{\gamma\in\Gamma} V_\gamma$, and so we can write $v=\sum v_\gamma$, $v_\gamma \in V_\gamma$. Using the fact that $V_\gamma A_0(Fx_k) \subset V_\gamma$ (see [1]), $v(x_i-\lambda_i 1)^{n_i}=0$, $i=1, \cdots, k-1$, implies $v_\gamma(x_i-\lambda_i 1)^{n_i}=0$, $i=1, \cdots, k-1$. Clearly for some γ, $v_\gamma \neq 0$, so defining $u=v_\gamma$ and $\lambda_n=\gamma(x_k)$ we have $u(x_i-\lambda_i 1)^{n_i}=0$ for $i=1, \cdots, k$. Define $\lambda: L \rightarrow F$ by $\lambda(x_i)=\lambda_i$ and extend to L by linearity. Then a straightforward calculation shows that $u(x-\lambda(x)1)^M=0$, $M=\sum_{i=1}^k n_i$, for any $x \in L$. Thus V is L-weighted.

LEMMA 1. *Let L be a diagonable subspace of a prime algebra A over F which possesses only a finite set Δ of roots. Suppose $x \in L$ is algebraic and X is any subalgebra of A containing x. Then if any irreducible X-module is x-weighted, every irreducible X-module is x-weighted.*

PROOF. It is sufficient to prove that the minimal (monic) polynomial $p(t)$ over F which x satisfies has all its roots in F. The key step towards this is establishing that $p(t)$ has the form

$$p(t) = \prod_{\alpha\in S} q(t+\alpha) \tag{3}$$

where $q(t) \in F[t]$ is irreducible and $S \subset \mathfrak{A}=\{\alpha(x):\alpha \text{ a root of } Fx \text{ in } A\}$. Since A is semiprime, so is the centralizer $A_0(x)$ of x by [1]. Thus $p(t)$ must be of the form $p_1(t) \cdots p_s(t)$, where $p_1(t), \cdots, p_s(t)$ are the distinct monic irreducible factors of $p(t)$ in $F[t]$. Let A_{ij} denote the subspace $\{a \in A: p_i(x)a=0=ap_j(x)\}$ for $1 \leqq i, j \leqq s$. Then we have

$$A = \bigoplus \sum_{i,j=1}^{s} A_{ij}. \tag{4}$$

To prove this, we note that the linear transformation $L_x: a \mapsto xa$ of A is algebraic with minimal polynomial also $p(t)$ (all algebras we consider contain 1). Thus $A=\bigoplus\sum_{i=1}^s A_i$ where $A_i=\{a \in A: p_i(x)a=0\}$. Now each subspace A_i is invariant under the linear transformation $R_x: a \mapsto ax$, which is also algebraic with minimal polynomial $p(t)$. Thus the restriction of R_x to A_i has minimal polynomial dividing $p(t)$, and so each A_i decomposes into a direct sum of spaces A_{ij} for some integers j, $1 \leqq j \leqq s$. This gives (4).

Now for each k, $1 \leqq k \leqq s$, define $a_k = \prod_{i \neq k} p_i(x)$. Then $a_k \neq 0$, and for any $i, j \in \{1, \cdots, s\}$, $a_i A a_j \neq 0$ because A is prime. But $a_i A a_j \subset A_{ij}$ and so each of the spaces A_{ij} is nonzero. Suppose then that $0 \neq a \in A_{j1}$ and $a = \sum_{\alpha \in \mathfrak{A}} a_\alpha$ is the decomposition of a relative to Fx (i.e. $A = \bigoplus \sum_{\alpha \in \mathfrak{A}} A_\alpha(x)$). For some α, $a_\alpha \neq 0$, and because $A_\alpha(x) A_0(x) \subset A_\alpha(x)$ and the sum $\sum_{\alpha \in \mathfrak{A}} A_\alpha(x)$ is direct, $a_\alpha p_1(x) = 0$. But $(a_\alpha, x) = \alpha a_\alpha$ easily implies $p_1(x+\alpha) a_\alpha = 0$. We have also $p_j(x) a = 0$ and so just as above $p_j(x) a_\alpha = 0$. Thus the polynomials $p_1(t+\alpha)$ and $p_j(t)$ cannot be relatively prime, and because they are irreducible and monic, $p_j(t) = p_1(t+\alpha)$. This establishes (3), where $q(t) = p_1(t)$.

Now by hypothesis, some irreducible X-module V is x-weighted, and the existence of a nonzero $v \in V$ and $\lambda \in F$ with $v(x - \lambda 1)^n = 0$ certainly implies $(x - \lambda 1)$ is not invertible in X. Thus the polynomials $p(t)$ and $t - \lambda$ are not relatively prime; i.e. $t - \lambda$ divides $p(t)$. It follows that for some $\alpha \in S$, $q(t+\alpha) = t - \lambda$; hence $q(t) = t - (\lambda + \alpha)$ and all roots of $p(t)$ lie in F.

This lemma, together with the proposition, gives one half of our theorem (because primitive algebras are prime). The other half is valid for L of arbitrary dimension.

Lemma 2. *Let A be a primitive algebra over F which decomposes $A = \bigoplus \sum_{\alpha \in \Delta} A_\alpha$ relative to a diagonable subspace L. Then if Δ is finite and all irreducible modules over A and $\mathscr{C}$ (the centralizer of L) are L-weighted, all elements of L are algebraic over F.*

Proof. We make use of Theorem 1.6 in [**1**] which states that $\mathscr{C}$ is a direct sum of finitely many primitive algebras R_i, $i = 1, \cdots, n$. If I is any maximal right ideal of $\mathscr{C}$, $\mathscr{C}/I$ is an irreducible $\mathscr{C}$-module and hence weighted; i.e. for some $\lambda : L \to F$ and $u \in \mathscr{C} \backslash I$, $u(x - \lambda(x) 1)^{n(x)} \in I$ for every $x \in L$. Since $x - \lambda(x) 1$ is in the centre of $\mathscr{C}$, it follows that $x - \lambda(x) 1 \in I$. For each i, there exists a maximal right ideal J_i of R_i containing no nonzero ideal of R_i. The above argument then shows that $J_i \oplus \sum_{j \neq i} R_j$, which is a maximal right ideal of $\mathscr{C}$, contains $x - \lambda_i(x) 1$ for every $x \in L$ and some $\lambda_i : L \to F$, $i = 1, \cdots, n$. For a particular $x \in L$, write $x = \sum_{i=1}^n x_i$, $x_i \in R_i$, and $1 = \sum_{i=1}^n e_i$, e_i the identity of R_i. By looking at the i-component of $x - \lambda_i(x) 1$, we see that $x_i - \lambda_i e_i \in J_i$, $\lambda_i = \lambda_i(x)$. But $x_i - \lambda_i e_i$ is in the centre of R_i because $x - \lambda(x) 1$ is in the centre of $\mathscr{C}$. Thus $(x_i - \lambda_i e_i) R_i$ is an ideal of R_i contained in J_i. It now follows that $x_i = \lambda_i e_i$, and $x = \sum_{i=1}^n \lambda_i e_i$ with $e_1, \cdots, e_n$ pairwise orthogonal idempotents. Since x satisfies the polynomial $\prod_{i=1}^n (t - \lambda_i) \in F[t]$, it is algebraic.

REFERENCES

1. E. G. Goodaire, *Irreducible representations of algebras*, Canad. J. Math. (to appear).

2. F. W. Lemire, *Weight spaces and irreducible representations of simple Lie algebras*, Proc. Amer. Math. Soc. **22** (1969), 192–197. MR **39** #4326.

3. ———, *Existence of weight space decompositions for irreducible representations of simple Lie algebras*, Canad. Math. Bull. **14** (1971), 113–115. MR **45** #328.

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