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A NOTE ON PERIODIC SOLUTIONS FOR DELAY-DIFFERENTIAL SYSTEMS[1]

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ABSTRACT. Let $f(t, x, y):[0, \infty)\times \boldsymbol{R}^n\times \boldsymbol{R}^n\to \boldsymbol{R}^n$ be continuous and 1-periodic in t, $\tau(t):[0, \infty)\to[0, h]$ $(0<h\leqq 1)$ continuous and 1-periodic. A simple geometric condition (Theorem 1) is given for the existence of 1-periodic solutions $x(t)$ of the nonlinear delay-differential system $x'(t)=f(t, x(t), x(t-\tau(t)))$, with $x(t)$ in a given bounded convex open set G in $\boldsymbol{R}^n$. The addition of a Lipschitz condition in x and monotonicity in y allows one to calculate $x(t)$ by a monotone sequence of successive approximations (Theorem 2). Extensions to a more general functional differential equation $x'(t)=g(t, x(t), x_t)$ are given.

1. **Introduction.** Let $f: [0, \infty)\times \boldsymbol{R}^n\times \boldsymbol{R}^n\to \boldsymbol{R}^n$, $\tau: [0, \infty)\to[0, h]$ be continuous, 1-periodic in the variable t, and consider the first order nonlinear differential system

$$x'(t) = f(t, x(t), x(t - \tau(t))) \qquad (' = d/dt). \tag{1}$$

The purpose of this note is to extend the range of applicability, and remove complicated hypotheses, in certain results of Mikolajska [3] concerning the existence and successive approximation of periodic solutions of (1) with $\tau(t)\equiv h\equiv 1$. We consider separately the question of successive approximations (§4); the existence is established by the following

THEOREM 1. *Let G be a bounded convex open set in $\boldsymbol{R}^n$ containing* 0, *and and assume there is a function $N: \partial G\to \boldsymbol{R}^n-\{0\}$ satisfying*

$$N(x)\cdot x > 0 \quad \textit{for } x\in \partial G, \tag{2}$$

$$\bar{G} \subseteq \{y: N(x)\cdot(y - x) \leqq 0\} \quad \textit{for each } x\in \partial G. \tag{3}$$

Suppose $f: [0, \infty)\times \boldsymbol{R}^n\times \boldsymbol{R}^n\to \boldsymbol{R}^n$, $\tau: [0, \infty)\to[0, h]$ are continuous functions, 1*-periodic in the variable t and*

$$N(x)\cdot f(t, x, y) \textit{ is positive (negative) for all } x\in \partial G,\ y\in \bar{G},\ t\in [0, 1]. \tag{4}$$

Then equation (1) *has a* 1*-periodic solution $x(t)$ with values in G.*

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COROLLARY. *Let* $f: [0, \infty) \times \boldsymbol{R}^n \times \boldsymbol{R}^n$, $\tau: [0, \infty) \to [0, h]$ *be continuous, and* 1*-periodic in the variable* t $(0 < h \leqq 1)$. *Assume that there exists a number* $R > 0$ *such that*

$$x \cdot f(t, x, y) > 0 \quad \textit{for all } |x| = R,\ |y| \leqq R,\ t \in [0, 1], \tag{5}$$

or

$$x \cdot f(t, x, y) < 0 \quad \textit{for all } |x| = R,\ |y| \leqq R,\ t \in [0, 1]. \tag{5$'$}$$

Then equation (1) *has a* 1*-periodic solution with values in* $|x| < R$. (*In the sequel,* $|\cdot|$ *denotes the Euclidean norm in* $\boldsymbol{R}^n$, *and* $x \cdot y$ *is the usual inner product.*)

REMARK. As noted in the corollary, it is often convenient to choose the set G to be a sphere with center 0. For second order scalar equations, it is sometimes more natural for G to be a rectangle with barycenter 0, and one can easily construct examples where a triangle leads to the simplest calculations.

We remark that $N(x)$ is not required to be continuous (see Theorem 2 for an application).

Conditions (2), (3) simply say that $N(x)$ is an outer normal for G, while relation (5) says that the outer normal $N(x)$ and the field f are in the same (or opposite) half-space determined by the hyperplane $N(x) \cdot [y - x] = 0$.

The above theorem was motivated by results of Z. Mikolajska [3], where functions similar to Lyapunov functions were used to establish the existence of periodic solutions of a scalar equation (1), with $\tau(t) \equiv h$, and period 1 replaced by period h. In honesty, these equations are really not delay equations from the point of view of periodic solutions, because the delay can be transformed out of the problem by a change of variables.

In this note, we obtain a number of improvements on the existence results in [3]. First, we consider equations (1) that are truly delay equations, and point out how to extend the results to more general functional differential equations (see §3). Secondly, we note that the complicated conditions B and B* of Mikolajska [3] imply the simple condition (5) of the corollary above, hence existence in [3] follows from our theorem. Finally, our theorem gives a simple geometric condition for existence of periodic solutions of systems (rather than scalar equations), without assuming monotonicity of f, or that f is Lipschitzian. In addition, our result is obtained by a very short proof, in which we appeal to Brouwer degree calculations, and an elegant theorem of V. V. Strygin [5].

2. **Proof of Theorem 1.** Let us show that our theorem follows as a simple application of a theorem of V. V. Strygin [5]. In order to apply the result in [5], we first define a continuum of nested convex regions G_λ, $\lambda \geqq 0$, as follows. Given $x \notin \bar{G}$, let $\bar{x}$ be the positive constant multiple of x

on ∂G, and put

$$G_\lambda = \{x \notin \bar{G} : |x - \bar{x}| < \lambda |x|\} \cup \bar{G}, \qquad \lambda > 0,$$
$$G_0 \equiv G.$$

The function $f(t, x, y)$ is modified, for purposes of the proof, as follows. Let f_1 be defined by

$$\begin{aligned} f_1(t, x, y) &= f(t, x, y), \qquad x \in \bar{G}, \\ &= f(t, \bar{x}, y), \qquad x \notin \bar{G}. \end{aligned}$$

Then define the modification F_ε of the function f by the formula

$$\begin{aligned} f_\varepsilon(t, x, y) &= \varepsilon f_1(t, x, y), \qquad y \in \bar{G}, \\ &= \varepsilon f_1(t, x, \bar{y}), \qquad y \notin \bar{G}. \end{aligned}$$

We observe that F_ε is continuous in all its variables. The idea is to show that the modified equation

$$(*) \qquad x'(t) = F_\varepsilon(t, x(t), x(t - \tau(t))) \qquad (0 < \varepsilon \leqq 1)$$

has a 1-periodic solution $x(t)$ in G for $\varepsilon = 1$. It then follows that $x(t)$ is a 1-periodic solution of equation (1) with values in G.

It will be shown that all possible 1-periodic solutions of equation $(*)$ have values in G. Let $x(t)$ be a 1-periodic solution of $(*)$, and assume, in order to obtain a contradiction, that $x(t)$ belongs to ∂G_λ for some $\lambda \geqq 0$. Then there exists a maximal $\lambda \geqq 0$ for which this is true. Hence, there is a point t_0 with $x(t_0) \in \partial G_\lambda$, and $x(t) \in \bar{G}_\lambda$ for all other values of t. In particular, $x(t - \tau(t)) \in \bar{G}_\lambda$, so we may apply condition (4) to obtain for $x_0 = x(t_0)$, $y_0 = x(t_0 - \tau(t_0))$, the relation (assume the positive sign in (4); the reasoning is similar for the negative sign)

$$N(\bar{x}_0) \cdot F_\varepsilon(t_0, x_0, y_0) > 0.$$

Let us write $x(t_0 + h) = x(t_0) + \int_0^1 x'(t_0 + sh)h\, ds$, and choose $h > 0$ so small that $N(\bar{x}_0) \cdot F_\varepsilon(t, x(t), x(t - \tau(t))) > 0$ for $t_0 \leqq t < t_0 + h$. Then equation $(*)$ gives

$$(**) \qquad N(\bar{x}_0) \cdot [x(t_0 + h) - x(t_0)] > 0.$$

Since $z_0 \equiv x(t_0 + h) \in \bar{G}_\lambda$, and $x_0 \in G_\lambda$, it follows that

$$\mu = |z_0|^{-1} |z_0 - \bar{z}_0| \leqq \lambda = |x_0|^{-1} |x_0 - \bar{x}_0|,$$

or else $z_0 \in \bar{G}$. The case $z_0 \in \bar{G}$ is eliminated by relations $(**)$ and (3).

Indeed, $x_0=a\bar{x}_0$, $a\geqq 1$, gives

$$0 < N(\bar{x}_0)\cdot[z_0 - x_0] = aN(\bar{x}_0)\cdot[a^{-1}z_0 - \bar{x}_0],$$

but $a^{-1}z_0 \in \bar{G}$ by convexity, so (3) gives the reverse sign in this inequality. On the other hand, if $z_0 \notin \bar{G}$, then $z_0=b\bar{z}_0$, $b>1$. We see that $\mu=1-b^{-1}$, $\lambda=1-a^{-1}$, and therefore $\mu\leqq\lambda$ gives $b\leqq a$. But then

$$0 < N(\bar{x}_0)\cdot[z_0 - x_0] = aN(\bar{x}_0)\cdot[a^{-1}b\bar{z}_0 - \bar{x}_0].$$

However, $a^{-1}b\leqq 1$ implies $a^{-1}b\bar{z}_0 \in \bar{G}$, and by (3) the sign must be reversed. Therefore, in both cases, a contradiction is reached. This proves that all possible 1-periodic solutions of equation (*) have values in G.

Let us calculate the Brouwer degree $d(T, G, 0)$ of the mapping

$$Tx = \int_0^1 F_\varepsilon(t, x, x)\,dt, \qquad x \in \bar{G}, \quad 0 < \varepsilon \leqq 1.$$

First, $F_\varepsilon(t, x, x)=\varepsilon f(t, x, x)$ for $x \in \bar{G}$; therefore (if (4) is positive)

$$N(x)\cdot Tx > 0, \qquad x \in \partial G.$$

Secondly, we may apply the Poincaré-Bohl transformation

$$H(x, \lambda) = \lambda Tx + (1-\lambda)x, \qquad x \in \bar{G}, \quad 0 \leqq \lambda \leqq 1,$$

to the identity and T, observe that $N(x)\cdot H(x, t)>0$ on ∂G, $0\leqq t\leqq 1$, because of (2) and (4), and conclude, by invariance of Brouwer degree under homotopy [4], that the degree in question is unity, hence nonzero.

We may now apply the result of V. V. Strygin [5, p. 601] to equation (*) and obtain a 1-periodic solution $x(t)$ of the modified equation with $x(t) \in G$, $0\leqq t\leqq 1$, for $\varepsilon=1$. The definition of the modified equation shows that $x(t)$ is a 1-periodic solution of equation (1), and the proof is complete.

3. **Generalizations.** Instead of equation (1), we may also consider the more general functional equation

$$x'(t) = f(t, x(t), x_t), \tag{6}$$

where $f: [0, \infty)\times \boldsymbol{R}^n \times C([-h, 0]\to \boldsymbol{R}^n)\to \boldsymbol{R}^n$ is continuous, and $x_t \in C([-h, 0]\to \boldsymbol{R}^n)$ is given as usual by the formula $x_t(\theta)=x(t+\theta)$, $-h\leqq\theta\leqq 0$. The very same theorem is valid for equation (6), provided we assume that $0<h\leqq 1$, f is 1-periodic in t, relations (2), (3) hold, and (4) is replaced by

(4′) For all $x \in \partial G$, $y \in C([-h, 0] \to \bar{G})$, $t \in [0, 1]$, we have $N(x)\cdot f(t, x, y) > 0$ (or negative).

The proof of §2 extends to this setting with only minor modifications.

4. **Successive approximations.** It is shown by Z. Mikolajska [3] that equation (1) with $\tau(t)\equiv h$ has a periodic solution $x(t)$ which is the limit of a decreasing sequence of successive approximations. To do this, Mikolajska imposes a complicated hypothesis (hypothesis B or B*) involving a function $V(t, x)$ similar to a Lyapunov function, and, in addition, it is assumed that $f(t, x, y)$ is increasing in y, h-periodic in t, and Lipschitz continuous in x. We replace Mikolajska's conditions B and B* by a simple geometric condition, apply Theorem 1, and obtain the following result for systems, which is more tractable for special equations.

THEOREM 2. *Let* $g:[0, \infty)\times \boldsymbol{R}^n\times \boldsymbol{R}^n\to \boldsymbol{R}^n$, $\tau:[0, \infty)\to[0, h]$ $(0<h\leqq 1)$ *be continuous and* 1*-periodic in* t, *with* $g(t, x, y)$ *Lipschitz continuous in* x, $g_i(t, x, y)$ *is increasing with respect to* y_k, $k\neq i$, $1\leqq k\leqq n$, $1\leqq i\leqq n$, *and*

(i) $x_i g_i(t, x, y)>0$ *for* $|x_i|=R$, $\max_{1\leqq j\leqq n}|x_j|\leqq R$, $\max_{1\leqq j\leqq n}|y_j|\leqq R$, $i=1, 2, \cdots, n$, *or*

(i)′ *the reverse inequality holds in* (i).

Then there is a sequence $\{x^n(t)\}_{n=0}^{\infty}$ *of* 1*-periodic functions satisfying*

(ii) $-R<x_i^{n+1}(t)<x_i^n(t)<R$, $n\geqq 1$, $1\leqq i\leqq n$,

(iii) $x_i^0(t)\equiv R$, $1\leqq i\leqq n$, *and*

$$x'_{n+1}(t) = g(t, x_{n+1}(t), x_n(t-\tau(t))), \qquad n>0,$$

which converge uniformly to a 1*-periodic solution of*

$$x'(t) = g(t, x(t), x(t-\tau(t))).$$

PROOF. Let $x^0(t)$ be given by (iii), let G be the parallelopiped defined by $|x_i|<R$ $(1\leqq i\leqq n)$. For $x\in\partial G$, let $N(x)=(\varepsilon_1, \cdots, \varepsilon_n)$ where $\varepsilon_i=0$ if $|x_i|<R$, $\varepsilon_i=x_i$ if $|x_i|=R$, $i=1, \cdots, n$. Then relations (2), (3) are easily verified. It is easy to see that (i) or (i)′ implies (4), with f replaced by g. Let us apply Theorem 1 to $f(t, x, y)=g(t, x, x^0(t))$, where $x^0(t)$ is defined in (iii). Then there exists a periodic solution $x^1(t)$ of $x'(t)=g(t, x(t), x^0(t))$ with values in G, and this gives inequality (ii) for $n=0$.

To establish the existence of the sequence $\{x^n(t)\}$, we proceed by induction, armed with techniques similar to those in Lemma 2, p. 28, in Mikolajska [3]. For this purpose, one needs a vector comparison theorem (see for example, P. Hartman [2, Example 4.1(6), p. 28]), in order to show that the set

$$Z^* = \{x\in \boldsymbol{R}^n: -R\leqq x_i\leqq x_i^n(0), 1\leqq i\leqq n\}$$

is mapped back into itself by the mapping

$$T^*: x\to y(1, x), \qquad y'(t, x) = g(t, y(t, x), x^n(t)), \qquad y(0, x) = x.$$

The Lipschitz condition makes T^* continuous, and the Brouwer fixed point theorem [**4**, pp. 74, 96] produces a fixed point of T^*, say x_0. Then $x^{n+1}(t)=y(t, x_0)$ is the desired periodic solution, completing the induction.

We use an Arzela-Ascoli theorem argument together with Dini's theorem to complete the proof.

COROLLARY. *Let* $g:[0, \infty)\times \boldsymbol{R}^1\times \boldsymbol{R}^1\to \boldsymbol{R}^1$, $\tau:[0, \infty)\to[0, h]$ $(0<h<1)$ *be continuous,* 1*-periodic in* t, $g(t, x, y)$ *Lipschitz continuous in* x *and increasing in* y, *with*

$$g(t, R, y)g(t, -R, y) < 0, \qquad 0 \leqq t \leqq 1, \quad (y) \leqq R.$$

Then there is a decreasing sequence of 1*-periodic functions with values in* $(-R, R)$ *satisfying* $x'_{n+1}(t)=g(t, x_{n+1}(t), x_n(t))$, *with* $\{x_n\}$ *uniformly convergent to a* 1*-periodic solution of* $x'(t)=g(t, x(t), x(t-\tau(t)))$.

EXAMPLE. Let $f(t, y):[0, \infty)\times \boldsymbol{R}^1\to \boldsymbol{R}^1$, $a(t):[0, \infty)\to \boldsymbol{R}^1$, $\tau(t):[0, \infty)\to [0, h]$ $(0<h\leqq 1)$ be continuous and 1-periodic in t, with f increasing in y, $|a(t)|>0$, and

$$\lim_{R\to\infty} R^{-1}\cdot \sup\{|f(t, y)|:0 \leqq t \leqq 1, |y| \leqq R\} = 0.$$

Then a periodic solution of $x'(t)=a(t)x(t)+f(t, x(t-\tau(t)))$ can be computed by successive approximations as in the corollary.

REMARK. The theorem has an immediate extension to functional equations of the form $x'(t)=g(t, x(t), x_t)$, which the reader can easily supply. The appropriate differential inequality theorem needed in the proof can be modeled after the one in P. Hartman [**2**, Example 4.1(6), p. 28] using the techniques in S. G. Deo and G. S. Ladde [**1**, Theorem 1, p. 47].

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