Area of Bernstein-type polynomials

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Abstract. Bernstein polynomials in one variable are known to be total-variation diminishing when compared to the approximated function \( f \). Here we consider the two variable case and give a counterexample to show they are not area-diminishing. Sufficient conditions are then given on a continuous function \( f \) to insure convergence in area. A similar theorem is proved for Kantorovitch polynomials in the case \( f \) is summable.

We consider the two-dimensional Bernstein polynomials \( B_{n,m}f \), and the corresponding Kantorovitch polynomials \( K_{n,m}f \), for functions \( z = f(x,y) \) defined on the unit square \( Q \). Sufficient conditions are given to insure the convergence in area of these polynomials. In particular if \( f \) is summable and generalized absolutely continuous on \( Q \), then \( L K_{n,m}f \to \Phi f \) where \( L \) is Lebesgue area, and \( \Phi \) is the Cesari-Goffman generalized area; if \( f \) is continuous and ACT, with \( R \)-integrable Tonelli lengths, then \( L B_{n,m}f \to Lf \).

For any \( f \) defined on all of \( Q \),

\[
B_{n,m}f(x,y) = \sum_{r=0}^{n} \sum_{s=0}^{m} f \left( \frac{r}{n}, \frac{s}{m} \right) p_{n,r}(x)p_{m,s}(y)
\]

where \( p_{n,R}(t) = \binom{n}{R} t^R (1-t)^{n-R} \).

For summable \( f \) on \( Q \),

\[
K_{n,m}f(x,y) = \sum_{r=0}^{n} \sum_{s=0}^{m} I_{r,s} p_{n,r}(x)p_{m,s}(y)
\]

where

\[
I_{r,s} = (n+1)(m+1) \int_{r/(n+1)}^{(r+1)/(n+1)} \int_{s/(m+1)}^{(s+1)/(m+1)} f(\xi, \eta) \, d\xi \, d\eta.
\]

If \( f \) is continuous, \( B_{n,m}f \) and \( K_{n,m}f \) converge uniformly to \( f \). Although the behavior of \( B_{n,m}f \) for discontinuous functions is quite erratic,
e.g. [L, p. 28], and [PJ], we have

**Proposition 1.** If \( f \) is summable on \( Q \), \( K_{n,m} f \) converges in the \( L_1 \) sense to \( f \).

**Proof.** For all \( m, n, \int_0^1 \int_0^1 K_{n,m} f = \int_0^1 \int_0^1 f \) because \( \int_0^1 p_N(t) \, dt = 1/(N+1) \) for any \( N \) and \( R = 0, 1, \ldots, N \). Hence \( \| K_{n,m} f \|_1 \leq \| f \|_1 \). Choose a continuous \( h \) such that \( \| f - h \|_1 \leq \epsilon/3 \). Then

\[
\| f - K_{n,m} f \|_1 \leq \| f - h \|_1 + \| h - K_{n,m} h \|_1 + \| K_{n,m} h - K_{n,m} f \|_1 \\
\leq 2 \| f - h \|_1 + \| h - K_{n,m} h \|_1.
\]

Since \( h \) is continuous, the last term is also at most \( \epsilon/3 \) for large \( m \) and \( n \), which completes the proof.

Cesari and later Goffman have defined equivalent areas for summable functions on \( Q \). We give Goffman’s version \([GJ]\). Let

\[
\Phi f = \inf \lim \inf_{(p_t)} L(p_t)
\]

where \( p_t \) are quasilinear functions converging \( L_1 \) to \( f \) and the inf is taken over all such sequences of \( p_t \). \( \Phi \) is lower semicontinuous with respect to \( L_1 \) convergence and coincides with \( L \) for continuous \( f \).

If \( f(x, y) \) is continuous, the linear variation for fixed \( y \) is denoted by \( V_{y,f}(y) \); similarly \( V_{x,f}(x) \). Their Lebesgue integrals, the Tonelli variations are \( V_y f = \int_0^1 V_{y,f}(y) \, dy \) and \( V_x f = \int_0^1 V_{x,f}(x) \, dx \). Correspondingly for summable \( f(x, y) \), the linear generalized variations are \( \varphi_{x,f}(y) \) and \( \varphi_{y,f}(x) \) where variation in each case is computed only over points of linear approximate continuity. The generalized Tonelli variations are \( \varphi_{x,f} = \int_0^1 \varphi_{x}(y) \, dy \) and \( \varphi_{y,f} = \int_0^1 \varphi_{y}(x) \, dx \). For continuous \( f \) and \( g \),

\[
L(f + g) \leq Lf + V_{y,g} + V_{x,g}
\]

and for summable \( f \) and \( g \),

\[
\Phi(f + g) \leq \Phi f + \varphi_{y,g} + \varphi_{x,g}.
\]

A continuous \( f(x, y) \) is ACT if \( V_{x,f} \) and \( V_{y,f} \) are finite and \( f \) is absolutely continuous on almost all lines parallel to each coordinate axis. A summable \( f \) is said to be \( g \)-ACT if \( \varphi_{x,f} \) and \( \varphi_{y,f} \) are finite, and there exists an \( h \sim g \) such that \( h \) is absolutely continuous on almost all lines parallel to each coordinate axis. Functions of \( g \)-ACT type may be “essentially discontinuous” i.e. every \( h \sim g \) is nowhere continuous \([G2]\).

For finite valued \( f(x) \) on \([0, 1]\),

\[
B_n f(x) \equiv \sum_{r=0}^{n} f \left( \frac{r}{n} \right) p_{n,r}(x)
\]
and for summable \( f \),

\[
K_n f(x) \equiv \sum_{r=0}^{n} (n + 1) \left( \int_{r/(n+1)}^{(r+1)/(n+1)} f(\xi) \, d\xi \right) p_{n,r}(x).
\]

Let \( V \) be total variation, \( \varphi \) be variation over points of approximate continuity, \( l \) the Jordan length, and \( \lambda \) the length over points of approximate continuity. Then for all \( n \),

\[
\begin{align*}
(a) \quad & VB_n f \leq Vf, \\
(b) \quad & VK_n f \leq \varphi f, \\
(c) \quad & lB_n f \leq lf, \\
(d) \quad & \lambda K_n f \leq \lambda f.
\end{align*}
\]

Part (a) is in \([L]\); (b) is in \([P_2]\); (c) and (d) follow from (a) and (b) by an integral-geometric formula of Cauchy and Steinhaus \([P_2]\). In virtue of the lower semicontinuity of \( V \) and \( l \) with respect to uniform convergence, and of \( \varphi \) and \( \lambda \) with respect to \( L_1 \) convergence, all four functionals converge as \( n \to \infty \). It is thus reasonable to conjecture \( LB_{n,m} f \to Lf \) and \( LK_{n,m} f \to \Phi f \) as \( n, m \to \infty \) for appropriate classes of functions.

There is a major difference in the two variable case however. Construct a \( C^\infty \) “rounded spike” function \( f_\varepsilon \) on \( Q \) which vanishes off a circular neighborhood \( C_\varepsilon \) of \((\varepsilon, \varepsilon)\) and assumes the value 1 at \((\varepsilon, \varepsilon)\). By making the spike sufficiently thin, \( Lf_\varepsilon = 1 + \varepsilon \) for arbitrarily small positive \( \varepsilon \). On the other hand \( B_{2\varepsilon} f_\varepsilon = 4\varepsilon^2 (1-x)(1-y) \) and is independent of the base radius \( r_\varepsilon \) of the spike. Hence, though \( f_\varepsilon \in C^\infty \), \( LB_{2\varepsilon} f_\varepsilon > 1 + \varepsilon = Lf_\varepsilon \) for some \( \varepsilon \) in contrast to the relations (2). We now state the theorems.

**Theorem 1.** \( f \) is \( gACT \), then \( \lim_{n,m \to \infty} LK_{n,m} f = \Phi f \).

**Proof.** \( \Phi \) is lower-semicontinuous with respect to \( L_1 \) convergence, so by Proposition 1, \( \liminf_{n,m \to \infty} LK_{n,m} f \geq \Phi f \).

By (1b),

\[
\Phi f \leq \liminf_{n,m} LK_{n,m} f \leq \limsup_{n,m} LK_{n,m} f = \limsup_{n,m} \Phi K_{n,m} f
\]

\[
\leq \Phi f + \limsup_{n,m} \varphi_\varepsilon(K_{n,m} f - f) + \limsup_{n,m} \varphi_\varepsilon(K_{n,m} f - f).
\]

It will be sufficient then to show (say) \( \varphi_\varepsilon(K_{n,m} f - f) \to 0 \). Since \( f \) is \( gACT \), \( \partial f/\partial x \) is summable, where \( \partial f/\partial x \) is the partial derivative with sets of measure zero neglected in the difference quotient \([G_1]\). Pick \( h \) continuously differentiable on \( Q \) such that \( \| \partial f/\partial x - H \|_1 < \varepsilon/3 \); i.e. \( \varphi_\varepsilon(f - H) < \varepsilon/3 \) where \( H(x, y) = \int_0^y h(t, y) \, dt \). Thus

\[
\varphi_\varepsilon(K_{n,m} f - f) \leq \varphi_\varepsilon(f - H) + V_\varepsilon(H - K_{n,m} H) + V_\varepsilon(K_{n,m} H - K_{n,m} f).
\]

The first term is \( < \varepsilon/3 \), and so is the second for large \( n \) and \( m \) because \( \partial K_{n,m} H/\partial x \to \partial H/\partial x \), since \( H \) is \( C^1 \). The proof of this follows from showing \( |\partial K_{n,m} /\partial x| - |\partial B_{n,m} /\partial x| \) to be small, and then using the corresponding result for \( B_{n,m} \) which is proved in \([B]\).
For the third term, we need a lemma which holds for any summable function.

**Lemma.** For $F(x, y)$ summable on $Q$ and all $m$ and $n$, $V_x K_{n,m} F \leq \varphi_x F$ (and $V_y K_{n,m} F \leq \varphi_y F$).

**Proof.**

\[
V_x K_{n,m} F = \int_0^1 \int_0^1 \frac{\partial K_{n,m} F}{\partial x} \, dx \, dy
\]

\[
= n \int_0^1 \int_0^1 \sum_{r=0}^{m} \sum_{s=0}^{n-1} |I_{r+1,s} - I_{r,s}| p_{n-1,r}(x)p_{m,s}(y) \, dx \, dy
\]

\[
\leq n \sum_{r=0}^{m} \sum_{s=0}^{m} \int_0^1 |I_{r+1,s} - I_{r,s}| p_{n-1,r}(x)p_{m,s}(y) \, dx \, dy
\]

\[
= \frac{1}{m+1} \sum_{r=0}^{n-1} \sum_{s=0}^{m} |I_{r+1,s} - I_{r,s}|.
\]

But

\[
|I_{r+1,s} - I_{r,s}| \leq (m+1) \int_{s/(m+1)}^{(s+1)/(m+1)} (n+1)
\]

\[
\cdot \int_{(r+2)/(n+1)}^{(r+1)/(n+1)} F(\xi, \eta) \, d\xi \, d\eta - \int_{r/(n+1)}^{(r+2)/(n+1)} F(\xi, \eta) \, d\xi \, d\eta
\]

and so

\[
V_x K_{n,m} F \leq \int_0^1 (n+1) \sum_{r=0}^{n-1} \int_{(r+2)/(n+1)}^{(r+1)/(n+1)} F(\xi, \eta) \, d\xi \, d\eta
\]

\[
- \int_{r/(n+1)}^{(r+2)/(n+1)} F(\xi, \eta) \, d\xi \, d\eta
\]

(3)

For almost all $\eta \in [0, 1]$, $F(\xi, \eta)$ is a summable function of $\xi$. For these $\eta$, the expression inside the first integral is at most $\varphi_x F(\eta)$. The proof is essentially that of (2)(b). Thus the right hand side of (3) is at most $\int_0^1 \varphi_x F(\eta) \, d\eta = \varphi_x F$ which completes the proof.

Now let $F = H - f$. $F$ is summable, and so by the lemma

\[
V_x (K_{n,m} H - K_{n,m} f) = V_x (K_{n,m} (H - f)) \leq \varphi_x (H - f) < \varepsilon/3.
\]

Hence $\varphi_x (K_{n,m} f - f) < \varepsilon$ for large $n$ and $m$ which completes the proof of Theorem 1.

For the next theorem, set $l_x f = \int_0^1 l_x f(y) \, dy$ where $l_x f(y)$ is the Jordan length in the $x$-direction of a section at $y$. Similarly define $l_y f$.

**Theorem 2.** If $f$ is $ACT$ and $l_x f$ and $l_y f$ are $R$-integrable, then $\lim_{n,m \to \infty} LB_{n,m} f = L f$. 

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Proof. Since \( B_{n,m}f \to f \) uniformly, \( \lim \inf_{n,m \to \infty} LB_{n,m}f \geq Lf \). By (1a), it is sufficient to show as in Theorem 1, that (say) \( V_x(B_{n,m}f - f) \to 0 \). Let \( h \) and \( H \) be as in Theorem 1 with \( V_x(f - H) < \epsilon/4 \). Then

\[
V_x(B_{n,m}f - f) \leq V_x(f - H) + V_x(H - B_{n,m}H) + V_x(B_{n,m}(H - f)).
\]

The first term is at most \( \epsilon/4 \), as is the second for large \( n \) and \( m \), because \( (\partial B_{n,m}H/\partial x) \to (\partial H/\partial x) \) uniformly [B]. For the third term, it is necessary to show \( V_x(f - H) \) is \( R \)-integrable.

Since \( l_xf \) is \( R \)-integrable, \( l_xf(y) \) and hence \( V_xf(y) \) is bounded for \( y \in [0, 1] \). Since \( H \) is \( C^1 \), \( V_x(f - H)(y) \) is bounded. In addition, \( V_x(f - H)(y) \) is continuous almost everywhere. To see this, pick \( y_0 \) from the full measure set where simultaneously \( f(x, y_0) \) is absolutely continuous as a function of \( x \), and \( l_xf(y) \) is continuous as a function of \( y \). Consider a sequence \( y_n \to y_0 \), and correspondingly the \( l_xf(y_n) \) and \( l_xH(y_n) \). Since \( H \) is \( C^1 \), \( H(x, y_0) \) is an absolutely continuous function of \( x \). By theorems in [A-L], \( l_x(f - H)(y_n) \to l_x(f - H)(y_0) \) which implies \( V_x(f - H)(y_n) \to V_x(f - H)(y_0) \). Thus \( V_x(f - H)(y) \) is continuous at almost all \( y \) and is \( R \)-integrable.

For arbitrary \( F(x, y) \), a computation similar to the lemma shows

\[
V_xB_{n,m}F \leq \frac{1}{m + 1} \sum_{s=0}^{m} V_xF \left( \frac{s}{m} \right)
\]

for all \( n, m \). Thus

\[
V_xB_{n,m}(H - f) \leq \frac{1}{m + 1} \sum_{s=0}^{m} V_x(H - f) \left( \frac{s}{m} \right)
\]

which converges to \( V_x(H - f) \) by \( R \)-integrability of \( V_x(H - f)(y) \). Hence for large \( m \) and all \( n \), the right hand side of (4) is less than \( 2(\epsilon/4) = \epsilon/2 \). For the same \( m \) and \( n \),

\[
V_x(B_{n,m}f - f) \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon,
\]

and the same computation for \( y \) shows \( V_y(B_{n,m}f - f) \to 0 \). Therefore \( \lim \sup LB_{n,m}f \leq Lf \) which completes the proof.

References


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