ABSOLUTE CONTINUITY OF EIGENVECTORS OF TIME-VARYING OPERATORS

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Abstract. If \( K(t) \) is a compact, selfadjoint operator function of a real variable \( t \) with distinct eigenvalues at each \( t \), we show that the eigenvalues and eigenvectors are absolutely continuous and that \( \{K(t)\} \) is a commuting set provided that \( K(t) \) commutes with its time derivative \( K'(t) \) at each \( t \). The distinct eigenvalue condition is shown to be necessary.

1. Introduction. Let \( H \) be a real Hilbert space, i.e. the inner product on \( H \) is real valued. Let \( K(t):H\rightarrow H \) be a compact linear operator for each \( t \) in a closed real interval \( J \), and assume \( K(t) \) is absolutely continuous in the operator norm on \( J \). We will prove the following:

**Theorem 1.** Assume \( \lambda(t) \) is a distinct (at each \( t \in J \), absolutely continuous eigenvalue function of \( K(t) \) on \( J \), and let \( \varphi(t) \) be a corresponding normalized eigenvector function. Then \( \varphi(t) \) can be chosen absolutely continuous on \( J \).

This result can be used inductively in connection with the usual spectral theory for compact, selfadjoint operators to prove the following:

**Theorem 2.** Assume \( K(t) \) is selfadjoint and has only distinct, nonzero eigenvalues at each \( t \in J \). Then there is a sequence of absolutely continuous (on \( J \)) eigenvalue and (corresponding, normalized) eigenvector functions: \( \{\lambda_n(t), \varphi_n(t)\} \), which include the spectrum of \( K(t) \) at each \( t \), and

\[
K(t)x = \sum \lambda_n(t)\varphi_n(t)\langle \varphi_n(t), x \rangle \quad (x \in H \text{ and } t \in J).
\]

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1 This problem occurred in connection with the problem of factoring unbounded operator functions during my doctoral research at Boston University, 1971. I wish to thank Professor Marvin I. Freedman for helpful suggestions.

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The results concerning continuity of eigenvalues and eigenvectors have a natural application to perturbation methods. Also Theorem 2 may be used to prove:

**Theorem 3.** Let $K(t)$ be selfadjoint and have only distinct, nonzero eigenvalues in $J$. Assume $K(t)$ commutes with its derivative $K'(t)$ almost everywhere in $J$. Then $\{K(t) | t \in J\}$ is a commuting set.

All the theorems can be extended to corresponding theorems for unbounded operator functions which have a compact inverse for each $t \in J$. Theorems similar to 2 and 3 are proven for a square matrix function $K(t)$ which is not selfadjoint (and can have the eigenvalue 0) in A. Acker [1]. An example in §4 shows the distinct eigenvalue condition is necessary in each above theorem.

2. **The proof of Theorem 1.** The proof is simplified by agreeing to call an operator $A$ "$\delta$-positive" (for a specified $\delta > 0$) if $|Ax| \geq \delta |x|$ for all vectors $x$ orthogonal to the nullspace of $A$. One can check that any operator of the form: $K - \lambda I$, where $K$ is compact, $I$ is the identity, and $\lambda \neq 0$, is $\delta$-positive for some, possibly very small, positive $\delta$. If we set $A(t) = K(t) - \lambda(t)I$, then Theorem 1 can be restated as follows:

**Lemma.** For each $t \in J$, let the operator $A(t) : H \rightarrow H$ be $\delta(t)$-positive (where $\delta(t) > 0$) and have 0 as a distinct eigenvalue. Assume the operator function $A(t)$ is absolutely continuous on $J$ in the operator norm. Then the corresponding (to 0) normalized eigenvector function $\varphi(t)$ can be chosen absolutely continuous on $J$.

**Proof.** At each $t$, $\varphi(t)$ is unique only to a factor of $\pm 1$. (Complex multiples are not in $H$.) For a specified choice of $\varphi(t)$ and $t_0$, $t \in J$, let $p(t_0, t) = \langle \varphi(t_0), \varphi(t) \rangle \varphi(t_0)$ and $p_{\perp}(t_0, t) = \varphi(t) - p(t_0, t)$ so that $p_{\perp}(t_0, t)$ is orthogonal to $\varphi(t_0)$ and $|p(t_0, t)|^2 + |p_{\perp}(t_0, t)|^2 = 1$. One can check that

\[(1) \quad A(t_0)p_{\perp}(t_0, t) = [A(t_0) - A(t)]\varphi(t).
\]

Since $A(t_0)$ is $\delta(t_0)$-positive and $|\varphi(t)| = 1$ for all $t$, equation 1 implies (where $\| \cdot \|$ is the operator norm) that

\[(2) \quad \delta(t_0)|p_{\perp}(t_0, t)| \leq \|A(t) - A(t_0)\| \quad \text{for all } t.
\]

We conclude that $|p_{\perp}(t_0, t)| \rightarrow 0$, $|p(t_0, t)| \rightarrow 1$, and $|\langle \varphi(t_0), \varphi(t) \rangle| \rightarrow 1$ as $t \rightarrow t_0$. For any $t_0 \in J$, there is a relatively open interval $I(t_0)$ containing $t_0$ on which $\langle \varphi(t_0), \varphi(t) \rangle \neq 0$. Redefine $\varphi(t)$ at each $t \in I(t_0)$ by multiplication by $\pm 1$ so that $\langle \varphi(t_0), \varphi(t) \rangle$ is positive. Then $\langle \varphi(t_0), \varphi(t) \rangle \rightarrow 1$ and $\varphi(t) \rightarrow \varphi(t_0)$ as $t \rightarrow t_0$. Let $I^*(t_0)$ be a relatively open subinterval of $I(t_0)$ in which $\langle \varphi(t_0), \varphi(t) \rangle > \frac{1}{2}$ and $|\varphi(t_0) - \varphi(t)| < \frac{1}{2}$. It is easily seen that
the product \( \langle \varphi(t_1), \varphi(t_2) \rangle \) is positive whenever \( t_1 \) and \( t_2 \) are in \( I^*(t_0) \), and therefore the preceding argument shows that the redefined function \( \varphi(t) \) is continuous throughout \( I^*(t_0) \).

It is intuitive that if \( A(t) \) and \( \varphi(t) \) (which spans the nullspace) are continuous, then \( A(t) \) is uniformly \( \delta \)-positive on a sufficiently small interval \( I^{**}(t_0) \) about \( t_0 \). This is proven starting with the equation:

\[
A(t)\psi = [A(t) - A(t_0)]\psi + A(t_0)[\psi - \langle \psi, \varphi(t_0) \rangle \varphi(t_0)]
\]

which holds whenever \( \psi \) is orthogonal to \( \varphi(t) \).

One can show that \( |\varphi(t_1) - \varphi(t_2)| < \sqrt{2} \| \varphi \| \) for \( t_1, t_2 \in I^*(t_0) \). Use this in connection with equation 2 and the uniform \( \delta \)-positivity of \( A(t) \) to obtain (for a fixed positive \( \delta \))

\[
\delta |\varphi(t_1) - \varphi(t_2)| < \sqrt{2} \| A(t_1) - A(t_2) \| \quad \text{(whenever } t_1, t_2 \in I^{**}(t_0) \text{)}.
\]

This immediately shows \( \varphi(t) \) is absolutely continuous on \( I^{**}(t_0) \).

The result is easily extended to the (compact) interval \( J \).

3. The proof of Theorem 3. It will be shown that if \( K(t) \) commutes with \( K'(t) \) (written: \( K(t) \sim K'(t) \)) a.e. in \( J \), then the absolutely continuous eigenvector functions \( \varphi_n(t) \) of \( K(t) \) are all constant. Assume \( \varphi(t) \) is the eigenvector function corresponding to the eigenvalue function \( \lambda(t) \) and let \( A(t) = K(t) - \lambda(t)I \). Then \( A(t) \) exists a.e. and \( A(t) \sim A'(t) \) a.e. in \( J \). Since \( 0 \) is a distinct eigenvalue of \( A(t) \), we conclude that \( \varphi(t) \) is also an eigenvector of \( A'(t) \), i.e. \( A'(t)\varphi(t) = \alpha(t)\varphi(t) \) for a real function \( \alpha(t) \). From \( A(t)\varphi(t) = 0 \) we obtain \( A'(t)\varphi(t) + A(t)\varphi'(t) = 0 \). Therefore \( A(t)\varphi'(t) + \alpha(t)\varphi(t) = 0 \) a.e. The vector product of this equation with \( \varphi(t) \) shows that \( \alpha(t) = 0 \) a.e. in \( J \). Therefore \( \varphi(t) \) and \( \varphi'(t) \) are both eigenvectors of \( A(t) \) at the distinct eigenvalue \( 0 \), and it follows that: \( \varphi'(t) = c(t)\varphi(t) \) for a real function \( c(t) \). We find that \( c(t) = \langle \varphi(t), \varphi'(t) \rangle \), so \( c(t) \) is integrable, and, for \( t_0 \in J \), the differential equation has a unique absolutely continuous solution \( \varphi(t) = C(t)\varphi(t_0) \), where \( C(t) = \int_{t_0}^{t} c(t')dt' \). \( C(t) \) is real and continuous, and \( |C(t)| = 1 \). Therefore \( C(t) = 1 \) and \( \varphi(t) = \varphi(t_0) \) for \( t \in J \).

4. The distinct eigenvalue condition is necessary. This is shown by an example. Let \( K_+ \) and \( K_- \) be two square matrices, each with distinct eigenvalues, which do not commute. Define \( K(t) \) as follows:

\[
K(t) = I + t^2K_- \quad \text{when } t \leq 0,
\]

\[
= I + t^2K_+ \quad \text{when } t \geq 0.
\]

Then \( K(t) \) has the following properties: 1. \( K(t) \) and its eigenvalues are absolutely continuous on any finite interval, and the eigenvalues are distinct except at \( t = 0 \). 2. \( K'(t) \) exists and \( K(t) \sim K'(t) \) at all \( t \) including 0. 3. The (continuous) eigenvectors of \( K(t) \) are constant on \((-\infty, 0) \) and
on $(0, \infty)$ and are simply the eigenvectors, respectively, of $K_-$ and $K_+$. Thus they are not continuous across $0$. 4. If $t_1 < 0 < t_2$, then $K(t_1)$ and $K(t_2)$ do not commute.

**Reference**

1. A. Acker, *Stability results for linear systems involving a time varying unbounded operator*, Doctoral Dissertation, Boston University, 1972, Appendix B.

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