ABSOLUTE CONTINUITY OF EIGENVECTORS
OF TIME-VARYING OPERATORS

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Abstract. If $K(t)$ is a compact, selfadjoint operator function of a real variable $t$ with distinct eigenvalues at each $t$, we show that the eigenvalues and eigenvectors are absolutely continuous and that $\{K(t)\}$ is a commuting set provided that $K(t)$ commutes with its time derivative $K'(t)$ at each $t$. The distinct eigenvalue condition is shown to be necessary.

1. Introduction. Let $H$ be a real Hilbert space, i.e. the inner product on $H$ is real valued. Let $K(t): H \to H$ be a compact linear operator for each $t$ in a closed real interval $J$, and assume $K(t)$ is absolutely continuous in the operator norm on $J$. We will prove the following:

Theorem 1. Assume $\lambda(t)$ is a distinct (at each $t \in J$), absolutely continuous eigenvalue function of $K(t)$ on $J$, and let $\varphi(t)$ be a corresponding normalized eigenvector function. Then $\varphi(t)$ can be chosen absolutely continuous on $J$.

This result can be used inductively in connection with the usual spectral theory for compact, selfadjoint operators to prove the following:

Theorem 2. Assume $K(t)$ is selfadjoint and has only distinct, nonzero eigenvalues at each $t \in J$. Then there is a sequence of absolutely continuous (on $J$) eigenvalue and (corresponding, normalized) eigenvector functions: $\{\lambda_n(t), \varphi_n(t)\}$, which include the spectrum of $K(t)$ at each $t$, and

$$K(t)x = \sum \lambda_n(t)\varphi_n(t)\langle \varphi_n(t), x \rangle \quad (x \in H \text{ and } t \in J).$$

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1 This problem occurred in connection with the problem of factoring unbounded operator functions during my doctoral research at Boston University, 1971. I wish to thank Professor Marvin I. Freedman for helpful suggestions.

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The results concerning continuity of eigenvalues and eigenvectors have a natural application to perturbation methods. Also Theorem 2 may be used to prove:

**Theorem 3.** Let $K(t)$ be selfadjoint and have only distinct, nonzero eigenvalues in $I$. Assume $K(t)$ commutes with its derivative $K'(t)$ almost everywhere in $I$. Then \{$K(t)|t \in I$\} is a commuting set.

All the theorems can be extended to corresponding theorems for unbounded operator functions which have a compact inverse for each $t \in I$. Theorems similar to 2 and 3 are proven for a square matrix function $K(t)$ which is not selfadjoint (and can have the eigenvalue 0) in A. Acker [1]. An example in §4 shows the distinct eigenvalue condition is necessary in each above theorem.

2. The proof of Theorem 1. The proof is simplified by agreeing to call an operator $A$ "δ-positive" (for a specified $\delta > 0$) if $|Ax| \geq \delta|x|$ for all vectors $x$ orthogonal to the nullspace of $A$. One can check that any operator of the form: $K-\lambda I$, where $K$ is compact, $I$ the identity, and $\lambda \neq 0$, is δ-positive for some, possibly very small, positive $\delta$. If we set $A(t) = K(t) - \lambda(t)I$, then Theorem 1 can be restated as follows:

**Lemma.** For each $t \in I$, let the operator $A(t): H \to H$ be $\delta(t)$-positive (where $\delta(t) > 0$) and have 0 as a distinct eigenvalue. Assume the operator function $A(t)$ is absolutely continuous on $I$ in the operator norm. Then the corresponding (to 0) normalized eigenvector function $\varphi(t)$ can be chosen absolutely continuous on $I$.

**Proof.** At each $t$, $\varphi(t)$ is unique only to a factor of $\pm 1$. (Complex multiples are not in $H$.) For a specified choice of $\varphi(t)$ and $t_0$, $t \in I$, let $p(t_0, t) = \langle \varphi(t_0), \varphi(t) \rangle \varphi(t)$ and $p_\bot(t_0, t) = \varphi(t) - p(t_0, t)$ so that $p_\bot(t_0, t)$ is orthogonal to $\varphi(t_0)$ and $|p(t_0, t)|^2 + |p_\bot(t_0, t)|^2 = 1$. One can check that

$$A(t_0)p_\bot(t_0, t) = [A(t_0) - A(t)]\varphi(t).$$

Since $A(t_0)$ is $\delta(t_0)$-positive and $|\varphi(t)| = 1$ for all $t$, equation 1 implies (where $\| \cdot \|$ is the operator norm) that

$$\delta(t_0)|p_\bot(t_0, t)| \leq \|A(t) - A(t_0)\| \quad \text{for all } t.$$

We conclude that $|p_\bot(t_0, t)| \to 0$, $|p(t_0, t)| \to 1$, and $|\langle \varphi(t_0), \varphi(t) \rangle| \to 1$ as $t \to t_0$. For any $t_0 \in I$, there is a relatively open interval $I(t_0)$ containing $t_0$ on which $\langle \varphi(t_0), \varphi(t) \rangle \neq 0$. Redefine $\varphi(t)$ at each $t \in I(t_0)$ by multiplication by $\pm 1$ so that $\langle \varphi(t_0), \varphi(t) \rangle$ is positive. Then $\langle \varphi(t_0), \varphi(t) \rangle \to 1$ and $\varphi(t) \to \varphi(t_0)$ as $t \to t_0$. Let $I^*(t_0)$ be a relatively open subinterval of $I(t_0)$ in which $\langle \varphi(t_0), \varphi(t) \rangle > \frac{1}{2}$ and $|\varphi(t_0) - \varphi(t)| < \frac{1}{2}$. It is easily seen that
the product $\langle \varphi(t_1), \varphi(t_2) \rangle$ is positive whenever $t_1$ and $t_2$ are in $I^*(t_0)$, and therefore the preceding argument shows that the redefined function $\varphi(t)$ is continuous throughout $I^*(t_0)$.

It is intuitive that if $A(t)$ and $\varphi(t)$ (which spans the nullspace) are continuous, then $A(t)$ is uniformly $\delta$-positive on a sufficiently small interval $I^{**}(t_0)$ about $t_0$. This is proven starting with the equation:

$$A(t)\psi = [A(t) - A(t_0)]\psi + A(t_0)[\psi - \langle \psi, \varphi(t_0) - \varphi(t) \rangle\varphi(t_0)]$$

which holds whenever $\psi$ is orthogonal to $\varphi(t)$.

One can show that $|\varphi(t_1) - \varphi(t_2)| < \sqrt{2} |p_{\perp}(t_1, t_2)|$ for $t_1, t_2 \in I^*(t_0)$. Use this in connection with equation 2 and the uniform $\delta$-positivity of $A(t)$ to obtain (for a fixed positive $\delta$)

$$\delta |\varphi(t_1) - \varphi(t_2)| < \sqrt{2} \|A(t_1) - A(t_2)\| \quad \text{(whenever } t_1, t_2 \in I^{**}(t_0)).$$

This immediately shows $\varphi(t)$ is absolutely continuous on $I^{**}(t_0)$.

The result is easily extended to the (compact) interval $J$.

3. The proof of Theorem 3. It will be shown that if $K(t)$ commutes with $K'(t)$ (written: $K(t) \sim K'(t)$) a.e. in $J$, then the absolutely continuous eigenvector functions $\varphi_n(t)$ of $K(t)$ are all constant. Assume $\varphi(t)$ is the eigenvector function corresponding to the eigenvalue function $\lambda(t)$ and let $A(t) = K(t) - \lambda(t)I$. Then $A'(t)$ exists a.e. and $A(t) \sim A'(t)$ a.e. in $J$. Since 0 is a distinct eigenvalue of $A(t)$, we conclude that $\varphi(t)$ is also an eigenvector of $A'(t)$, i.e. $A'(t)\varphi(t) = \alpha(t)\varphi(t)$ for a real function $\alpha(t)$.

From $A(t)\varphi(t) = 0$ we obtain $A'(t)\varphi(t) + A(t)\varphi'(t) = 0$. Therefore $A(t)\varphi'(t) + \alpha(t)\varphi(t) = 0$ a.e. The vector product of this equation with $\varphi(t)$ shows that $\alpha(t) = 0$ a.e. in $J$. Therefore $\varphi(t)$ and $\varphi'(t)$ are both eigenvectors of $A(t)$ at the distinct eigenvalue 0, and it follows that: $\varphi'(t) = c(t)\varphi(t)$ for a real function $c(t)$. We find that $c(t) = \langle \varphi(t), \varphi'(t) \rangle$, so $c(t)$ is integrable, and, for $t_0 \in J$, the differential equation has a unique absolutely continuous solution $\varphi(t) = C(t)\varphi(t_0)$, where $C(t) = \int_{t_0}^{t} c(t') \, dt'$. $C(t)$ is real and continuous, and $|C(t)| = 1$. Therefore $C(t) = 1$ and $\varphi(t) = \varphi(t_0)$ for $t \in J$.

4. The distinct eigenvalue condition is necessary. This is shown by an example. Let $K_+ \text{ and } K_-$ be two square matrices, each with distinct eigenvalues, which do not commute. Define $K(t)$ as follows:

$$K(t) = I + t^2K_- \quad \text{when } t \leq 0,$$
$$= I + t^2K_+ \quad \text{when } t \geq 0.$$ 

Then $K(t)$ has the following properties: 1. $K(t)$ and its eigenvalues are absolutely continuous on any finite interval, and the eigenvalues are distinct except at $t=0$. 2. $K'(t)$ exists and $K(t) \sim K'(t)$ at all $t$ including 0. 3. The (continuous) eigenvectors of $K(t)$ are constant on $(-\infty, 0)$ and
on \((0, \infty)\) and are simply the eigenvectors, respectively, of \(K_-\) and \(K_+\). Thus they are not continuous across 0. If \(t_1 < 0 < t_2\), then \(K(t_1)\) and \(K(t_2)\) do not commute.

**Reference**

1. A. Acker, *Stability results for linear systems involving a time varying unbounded operator*, Doctoral Dissertation, Boston University, 1972, Appendix B.

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