CONTINUITY OF CERTAIN CONNECTED FUNCTIONS AND MULTIFUNCTIONS

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Abstract. In this paper it is proved that if $X$ is a 1st countable, locally connected, $T_1$-space and $Y$ is a $\sigma$-coherent, sequentially compact $T_1$-space, then any nonmingled connectedness preserving multifunction from $X$ onto $Y$ with closed point values and connected inverse point values is upper semicontinuous. It follows that any monotone, connected, single-valued function from $X$ onto $Y$ is continuous. Let $X$ be as above and let $Y$ be a sequentially compact $T_1$-space with the property that if a descending sequence of connected sets has a nondegenerate intersection, then this intersection must contain at least three points. If $f$ is a monotone connected single-valued function from $X$ onto $Y$, then $f$ is continuous. An example of a noncontinuous monotone connected function from a locally connected metric continuum onto an hereditarily locally connected metric continuum is given.

In [1] and [2] conditions are given under which an open monotone connected function is continuous. This paper is concerned with conditions under which a monotone connected function is continuous. As Example 2 below shows, a monotone connected function from an hereditarily locally connected metric continuum onto a nonlocally connected metric continuum is not necessarily continuous, and Example 3 shows that a monotone connected function from a locally connected metric continuum onto an hereditarily locally connected metric continuum is not necessarily continuous. It is an open question as to whether or not such a function is continuous if both the domain and range are hereditarily locally connected metric continua.

Some definitions will now be recalled. A multifunction $F: X \rightarrow Y$ is upper semicontinuous at a point $p \in X$ if for any open set $V \subseteq Y$, with $F(p) \subseteq V$, there is an open set $U \subseteq X$, with $p \in U$, such that $F(U) \subseteq V$, and $F$ is nonmingled provided for any $p, q \in X$, either $F(p) = F(q)$ or $F(p) \cap F(q) = \emptyset$.

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See [6] for other terminology and properties of multifunctions. A single-valued function is monotone if point inverses are connected and is connected if the function preserves connectedness. This same terminology will be used for multifunctions also. A space $Y$ is $\sigma$-coherent provided any descending sequence of connected sets has a connected intersection. Finally, lim sup $A_n$ will denote the upper limit of a sequence $\{A_n\}$ of sets as defined on p. 337 of [3].

**Theorem 1.** Let $X$ be a 1st countable, locally connected $T_1$-space and $Y$ a $\sigma$-coherent, sequentially compact $T_\infty$-space. If $F$ is a nonmingled connected multifunction from $X$ onto $Y$ with closed point values, such that $F^{-1}$ has connected point values, then $F$ is upper semicontinuous.

**Proof.** Suppose $F$ is not upper semicontinuous at the point $p \in X$. Then there is an open set $V \subseteq Y$ such that $F(p) \subseteq V$, but for any open set $U \subseteq X$, with $p \in U$, there is some $x \in U$ such that $F(x)$ is not contained in $V$. Let $\{U_n\}$ be a countable base at $p$ consisting of open connected sets with $U_{n+1} \subseteq U_n$ for all $n$. For each $n$, let $p_n \in U_n$ such that $F(p_n)$ is not contained in $V$, and let $A_n = (Y-V) \cap F(p_n)$. Then $\{A_n\}$ is a sequence of sets all lying in the closed set $Y-V$. Since $Y$ is sequentially compact, there is a point $q \in (Y-V) \cap \lim \sup A_n$. If $q \in F(p_n)$ for all but finitely many $n$, then it can be assumed that $q \in F(p_n)$ for all $n$. Thus, $F(p_1) = F(p_n)$ for all $n$, since $F$ is nonmingled. Therefore, $F(p_n) \subseteq F(U_n)$ for all $n$, which implies that $q \in K = \bigcap_{n=1}^{\infty} F(U_n)$. Also, $F(p) \subseteq V$ and $q \in (Y-V)$. Hence, $F(p)$ and $q$ are separated in $K$ since $Y$ is $\sigma$-coherent. Therefore, there is a point $y$ in $K-\{F(p) \cup q\}$. Since $y \in F(U_n)$ for all $n$, $F^{-1}(y) \cap U_n \neq \emptyset$ for all $n$. Therefore $p$ is a limit point of $F^{-1}(y)$. But by Corollary D2 of [6], $F^{-1}(y)$ is closed. Hence, $p \in F^{-1}(y)$. This implies $y \in F(p)$, which is a contradiction. Thus, it must be the case that $q \notin F(p_n)$ for infinitely many $n$. Now $F(p_j) \subseteq F(U_n)$ for all $j \geq n$, and every neighborhood of $q$ intersects $F(p_j)$ for infinitely many $j$. Thus, $q$ is a limit point of $F(U_n)$ for every $n$. Therefore, $\{F(U_n) \cup q\}$ is a descending sequence of connected sets, and by hypothesis $K = \bigcap_{n=1}^{\infty} (F(U_n) \cup q)$ is connected. But again, $F(p)$ and $q$ are separated in $K$ and therefore there is a point $y$ in $K-\{F(p) \cup q\}$. This leads to the same contradiction as before. Hence, $F$ must be upper semicontinuous.

**Corollary.** If $X$ and $Y$ are as in Theorem 1 and $f$ is a monotone, connected, single-valued function from $X$ onto $Y$, then $f$ is continuous.

**Proof.** A single-valued function is nonmingled, and since $Y$ is $T_1$, $f$ has closed point values. Also, $f$ monotone means $f^{-1}$ has connected point values.
Theorem 2. Let $X$ be as in Theorem 1 and let $Y$ be a sequentially compact $T_1$-space with the property that if a descending sequence of connected sets has a nondegenerate intersection, then this intersection must contain at least three points. If $f$ is a monotone connected single-valued function from $X$ onto $Y$, then $f$ is continuous.

Proof. The proof proceeds exactly as in the proof of Theorem 1. The set $K$ has at least two points $f(p)$ and $q$, so by hypothesis must contain a third point $y$ distinct from $f(p)$ and $q$. This leads to the same contradiction as in the proof of Theorem 1. Thus, $f$ must be upper semicontinuous. But this is equivalent to continuity since $f$ is single-valued.

The following example shows that not every hereditarily locally connected metric continuum has the property given in the hypothesis of Theorem 2, that if a descending sequence of connected sets has a nondegenerate intersection, then the intersection has at least three points.

Example 1. This is a modification of an example given in [4, p. 284]. In the plane let $C_{nk}$ denote the semicircle given by

\[(x - (2k - 1)/2^n)^2 + y^2 = 1/4^n, \quad y \geq 0.\]

Denote by $L_{nk}$ the straight line segment given by $x = (2k - 1)/2^n, 0 \leq y \leq 1/2^n$. Let $Q_{nk}$ denote the semicircle given by

\[(x - (2k - 1)/(2 \cdot 3^n))^2 + y^2 = 1/(4 \cdot 9^n), \quad y \leq 0.\]

Denote by $R_{nk}$ the straight line segment given by

\[x = (2k - 1)/(2 \cdot 3^n), \quad -(2k - 1)/(2 \cdot 3^n) \leq y \leq 0.\]

Let $H_n$ denote the union of all the $C_{nk}$ and $L_{nk}$, $1 \leq k \leq 2^{n-1}$, and denote by $K_n$ the union of all the $Q_{nk}$ and $R_{nk}$, $1 \leq k \leq 3^n$. Finally, let $X$ denote the union of all the $H_n$ and all the $K_n$, $n$ varying over all positive integers, along with the interval $I = [0, 1]$. Then $X$ is an hereditarily locally connected continuum. Let $D$ denote the set of end points of all upper semicircles $C_{nk}$ and $T$ the set of end points of all lower semicircles $Q_{nk}$, and let $p = (0, 0)$ and $q = (1, 0)$. Let $U_1 = (X - I) \cup D \cup T$ and for $n \geq 1$, let $U_{n+1} = (U_n - (H_n \cup K_n)) \cup \{p, q\}$. Then $\{U_n\}$ is a descending sequence of connected sets whose intersection consists of just the points $p$ and $q$.

In the following example, an hereditarily locally connected $\sigma$-coherent continuum is mapped by a noncontinuous one-to-one connected function $f$ onto a nonlocally connected, 1st countable continuum. The function $f^{-1}$ is also connected and noncontinuous. Since all the hypotheses of Theorem 1 is satisfied for $f^{-1}$ except local connectedness of the domain of $f^{-1}$, this property is necessary in Theorem 1.
Example 2. Choose a polar coordinate system on the plane and for each positive integer \( n \), let \( L_n \) denote the segment \( \{(r, (\pi/2)/n) | 0 \leq r \leq 1/n\} \). Let \( X = \bigcup_{n=1}^\infty L_n \). Now choose a rectangular coordinate system on the plane with the same origin, and for each positive integer \( n \), let \( S_n \) denote the segment \( \{(x, y) | 0 \leq x \leq 1, y = x/n\} \), and \( S_0 \) the segment \( \{(x, 0) | 0 \leq x \leq 1\} \). Then \( Y = \bigcup_{n=0}^\infty S_n \) is a nonlocally connected non-\( \sigma \)-coherent metric continuum. Define a function \( f \) from \( X \) onto \( Y \) as follows: For each \( n \geq 1 \), let \( p_n = (1/n, (\pi/2)/n) \) in polar coordinates, and \( q_n = (1, 1/n) \) in rectangular coordinates. Let \( p_0 = (0, 0) \) and \( q_0 = (1, 0) \). Let \( f \) be the function that takes \( L_n \) linearly onto \( S_{n-1} \) such that \( f(p_0) = p_0 \) and \( f(p_n) = q_{n-1}, n \geq 1 \). Then \( f \) is one-to-one, connected, and not continuous at \( p_0 \). Also, the function \( f^{-1} \) taking \( Y \) onto \( X \) satisfies all the hypothesis of Theorem 1 except that the domain \( Y \) is not locally connected, and \( f^{-1} \) is not continuous at \( q_0 \). Thus, local connectedness of the domain space is necessary in Theorem 1.

Finally, the following is an example of a noncontinuous, monotone, connected function from a locally connected metric continuum onto an hereditarily locally connected metric continuum.

Example 3. Choose a rectangular coordinate system on Euclidean 3-space \( E_3 \) and let \( p = (0, 0, 0) \) and \( q = (1, 0, 0) \). Also, choose a spherical coordinate system, as defined on p. 355 of [5], with origin \( p \). Thus \( q = (1, 0, \pi/2) \) in spherical coordinates. For each \( n \), let

\[
p_n = (1/n, \pi/2, \pi/(2n)) \quad \text{and} \quad q_n = \left((2^n - 1)/2^n, 0, \pi/2\right)
\]

in spherical coordinates. If \( a \) and \( b \) are end points of a straight line segment, let \( ab \) denote the ordered segment from \( a \) to \( b \). Let \( P_n \) denote the plane determined by the three points \( p, p_n, q_n \), \( L_n \) the ordered segment from \( p_n \) to \( q_n \), and \( S_n \) the ordered segment from \( p \) to \( p_n \).

Define inductively a sequence of finite ordered subsets of \( pq \) as follows:

Let \( H_2 = \{x_{21}, x_{22}\} \), where \( x_{21} \) and \( x_{22} \) are the mid points, respectively, of the segments \( pq_1 \) and \( q_1q_2 \). Let \( H_3 = \{x_{31}, x_{32}, x_{33}, x_{34}, x_{35}\} \), where these points are, respectively, the mid points of the segments \( px_{21}, x_{21}q_1, q_1x_{22}, x_{22}q_2, q_2q_3 \). Assuming \( H_{n-1} \) has been defined, let \( H_n = \{x_{n1}, \ldots, x_{nk_n}\} \), where the \( x_{ni} \) are the mid points of the ordered collection of segments of \( pq_n \) determined by the points of \( H_{n-1} \). The order of the listing in \( H_n \) is by increasing distance from \( p \), where \( x_{n1} \) is the mid point of the segment \( px_{n-1,1} \) and \( x_{nk_n} \) is the mid point of the segment \( q_{n-1}q_n \). Note that the union of the \( H_n \)'s is a countable dense subset of \( pq \).

For each \( n \geq 2 \) and each \( j, 1 \leq j \leq k_n \), let \( L_{nj} \) denote the line segment parallel to \( L_n \) with one end point at \( x_{nj} \) and the other end point, denoted by \( s_{nj} \), on the segment \( S_n \). Thus, \( \{s_{n1}, \ldots, s_{nk_n}\} \) is a finite subset of \( S_n \) and each \( s_{nj} \) is joined to the corresponding \( x_{nj} \) by the segment \( L_{nj} \) lying in the plane \( P_n \) and parallel to \( L_n \).
Let \( K_n = (\bigcup_{i=1}^{n} H_i) \cup \{q_1, \ldots, q_{n-1}\} \). For each point \( y \in K_n \), let \( L_y \) denote the segment parallel to \( S_n \) with one end point at \( y \) and the other end point on \( L_n \). Thus, each \( L_y \) lies in the plane \( P_n \). Let \( Q_1 = pq \cup pp_1 \cup p,g_1 \), and for \( n \geq 2 \), let \( Q_n \) denote the union of the segments \( pq, S_n, L_n, L_y, y \in K_n \), and \( L_n, 1 \leq j \leq k_n \). Then \( X = \bigcup_{n=1}^{\infty} Q_n \) is a locally connected continuum.

Remark. Let \( R_n \) denote the plane given by \( z = y/n \) in rectangular coordinates, where \( n \) is a positive integer. It is to be understood that any arc of a circle subsequently described with end points on \( pq \) and lying in \( R_n \) is to have altitude less than one-half of the minimum of the altitudes of all such previously described arcs and is disjoint from all such arcs except possibly at some end points.

\[ \text{Figure 1} \]
Define a function $f$ on $X$ into $E_3$ as follows: It will be sufficient to define $f$ on each $Q_n$. Note that $Q_1 = pq \cup pp_1 \cup p q_1$. The function $f$ will be the identity function on $pq$. Let $f$ take $pp_1$ homeomorphically onto an arc of a circle of radius one-half the distance from $p$ to $q_1$ lying in $R_1$ with end points $p$ and $q_1$, with $f(p_1) = q_1$, and let $f(p, q_1) = q_1$. The function $f$ will now be defined on $Q_2$ and it will be clear that the same process can be used on any $Q_n$. The set $K_3$ consists of the points $x_{31}, x_{21}, x_{32}, q_1, x_{33}, x_{22}, x_{34}, q_2, x_{35}$, and $Q_3$ is

$$pq \cup S_3 \cup L_3 \cup \left( \bigcup_{i=1}^{3} L_{3i} \right) \cup \left( \bigcup \{ L_y \mid y \in K_3 \} \right).$$

Define $f$ on $S_3$, which is $ps_{31} \cup s_{31}s_{32} \cup s_{32}s_{33} \cup s_{33}s_{34} \cup s_{34}s_{35} \cup s_{35}p_3$, as follows: Let $f$ take $ps_{31}$ homeomorphically onto an arc of a circle lying in $R_3$ where the arc has end points $p$ and $x_{31}$ and altitude one-half the distance from $p$ to $x_{31}$, and $f(s_{31}) = x_{31}$. Let $f$ take $s_{31}s_{32}$ homeomorphically onto an arc of a circle lying in $R_3$ where the arc has end points $x_{31}, x_{32}$ and is subject to the conditions in the above remark, and $f(s_{32}) = x_{32}$. Map segments $s_{32}s_{33}, s_{33}s_{34}, s_{34}s_{35}, s_{35}p_3$ in a similar manner. Let $f$ take $s_{33}p_3$ homeomorphically onto an arc of a circle lying in $R_3$ where the arc has end points $x_{35}, q_3$ and subject to the conditions in the above remark, and $f(s_{33}) = x_{35}$ and $f(p_3) = q_3$. Let $f(L_3) = q_3$ and $f(L_3) = x_{31}, 1 \leq i \leq 5$.

For $y = x_{31} \in K_3$, $L_y$ intersects $L_{3j}$ at, say $z_{1j}$, $1 \leq j \leq 5$, where $z_{11} = x_{31}$, and $L_y$ intersects $L_3$ at, say $z_3$. Map $x_{31}z_{12}$ homeomorphically onto an arc of a circle lying in $R_3$ satisfying the conditions in the above remark, with end points $x_{31}, x_{32}$, where $f(z_{12}) = x_{32}$. Let $f$ take $z_{12}z_{13}$ homeomorphically onto an appropriate arc of a circle lying in $R_9$ with end points $x_{32}, x_{33}$, where $f(z_{13}) = x_{33}$. Map $z_{13}z_{14}$ homeomorphically onto an appropriate arc.

Figure 2
of a circle lying in $R_3$ with end points $x_{33}, x_{34}$, where $f(z_{15}) = x_{34}$. Map $z_{15}z_{16}$ in a similar manner. Let $f$ take $z_{15}z_{3}$ homeomorphically onto an appropriate arc of a circle lying in $R_3$ with end points $x_{35}, q_3$, where $f(z_{15}) = x_{35}$ and $f(z_{3}) = q_3$. This defines $f$ on $L_y$, where $y = x_{31}$. Define $f$ similarly on each $L_y$, $y \in K_3$, where the arcs chosen are subject to the conditions in the above remark. The function is now defined on $Q_3$. The set $Q_3$ and its image $f(Q_3)$ are displayed in Figures 1 and 2, respectively. In Figure 2 the curves are supposed to represent circular arcs. Define $f$ in a similar manner on all the $Q_n$, where the initial arcs are to be chosen in $R_n$ having radius one-half the distance from $p$ to $x_{n1}$, and subject to the conditions in the above remark.

The resulting function $f$ is monotone since the inverse of a point is either a point or a segment, $f$ maps connected sets onto connected sets, $f(X)$ is a hereditarily locally connected continuum in $E_3$, but $f$ is not continuous at $p$ since the sequence $\{p_n\}$ converges to $p$ and the sequence $\{f(p_n)\}$ converges to $q \neq f(p)$.

**BIBLIOGRAPHY**


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