THE M.H.D. VERSION OF A THEOREM OF H. WEYL

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ABSTRACT. In his discussion of shock waves in arbitrary fluids, H. Weyl proves a theorem concerning the behavior of the entropy function along the Hugoniot curve. The analogous result is proven for the M.H.D. case.

The theorem of the title concerns the behavior of a function on a curve: the function represents entropy, the curve is the Hugoniot curve. Such a curve is determined for each point \((V_0, p_0)\) in the positive quadrant of the (specific) volume-pressure plane. Weyl [3] shows for gas dynamics that the entropy function restricted to this curve has at most one critical point, and that \((V_0, p_0)\) is the candidate. The significance of the result is that the entropy behaves in the physically expected way across shocks. For the general context the reader is referred to [2]; here we only want to point out how Weyl's clever argument works in the magnetohydrodynamic (m.h.d.) case.

The Hugoniot curve in the \((V, p)\) plane corresponding to the point \((V_0, p_0)\) is defined [1] by the equation

\[
H(V, p) = e - e_0 + \frac{1}{2}(p + p_0)(V - V_0) + \phi(V) = 0.
\]

Here \(e = e(V, p)\) represents internal energy and \(e_0 = e(V_0, p_0)\). The function \(\phi(V)\) is identically zero in the case of gas dynamics while in the m.h.d. case \(\phi\) takes the form

\[
\phi(V) = K(V - V_0)^2(V - V_1)^{-2}
\]

where \(K\) and \(V_1\) \((\neq V_0)\) are positive constants (see [1]).

We use the following facts and hypothesis:

\[
de = -p \, dV + T \, ds, \text{ where } T \text{ and } s \text{ denote respectively the temperature and entropy.}
\]

\[
\text{Given } V \text{ and } s, \text{ a unique value } p = p(V, s) \text{ is determined. For the function so defined, } \partial p / \partial V^2 \text{ and } \partial p / \partial s \text{ are both positive.}
\]

\[
\phi'(V_0) = \phi'(V_0) = 0 \text{ and } \phi''(V) = V - V_0 \geq 0.
\]
For the $\phi$ of m.h.d. this is satisfied. In particular we compute

$$
\phi'(V) = 3K(V - V_0)^2(V - V_1)^{-2} - 2K(V - V_0)^3(V - V_1)^{-3},
$$

$$
\phi''(V) = 6K(V - V_0)(V - V_1)^{-2} - 12K(V - V_0)^2(V - V_1)^{-3}
+ 6K(V - V_0)^2(V - V_1)^{-4},
$$

$$
\phi'(V)(V - V_0) = 6K((V - V_0)(V - V_1)^{-1} - (V - V_0)^2(V - V_1)^{-2})^2
\geq 0.
$$

The main lemma (8) used to examine the behavior of $s$ on the set where $H=0$ concerns the behavior of $s$ on a different family of curves. These are defined to be those curves restricted to which the one-form $\omega = dH - T\, ds$ vanishes.

From (1), using (3), we compute

$$
dH = \left[ \frac{1}{2}(p_0 - p) + \phi'(V) \right] dV + \frac{1}{2}(V - V_0) dp + T\, ds.
$$

Thus the integral curves of $\omega = 0$ are solutions of the ordinary differential equation

$$
\dot{V} = V - V_0, \quad \dot{p} = p - p_0 - 2\phi'(V).
$$

Using $\phi'(V_0) = 0$ in (5), observe that the point $(V_0, p_0)$ is a repelling rest point of these equations; we let $R$ denote the set of points in the positive $(V, p)$ quadrant which are in the domain of repulsion.

Note that if $\phi \equiv 0$ then $R$ is the whole positive quadrant. Also with the $\phi$ of m.h.d., $R$ contains precisely those points of the quadrant which lie on the same side of the line $V = V_1$ as does $V_0$. Namely, given any such point $(V, p)$, the $V$ component of the solution goes to $V_0$ as $t \to -\infty$.

This, with $\phi'(V_0) = 0$, implies the $p$ component goes to $p_0$. Also, as the $V$ component tends to $V_1$, the absolute value of the $p$ component must go to infinity.

Now let $\gamma$ be any integral curve of (6) in $R$. The relevant facts about the behavior of $s$ on $\gamma$ are the following:

(7) The critical points of $H|\gamma$ and $s|\gamma$ coincide.

This follows from $dH = \omega + T\, ds.$

(8) Any critical point of $s|\gamma$ is a maximum; in particular, $s|\gamma$ has at most one critical point.

To see this we observe there are two ways of writing the derivative, $\dot{p}$, of $p$ on $\gamma$; namely,

$$
p - p_0 - 2\phi'(V) = \dot{p} = p_V\dot{V} + p_\delta.
$$
Now differentiate this equation again along $\gamma$ and evaluate at a critical point of $s$ ($\dot{s}=0$). Writing 0 for terms involving $\dot{s}$ we have

$$p_{\gamma}\dot{V} + 0 - 2\phi''(V)\dot{V} = p_{\gamma}\dot{V}^2 + 0 + p_{\gamma}\dot{V} + 0 + p_{\gamma}\dot{s}.$$ 

From (6) we see that $\dot{V} = V - V_0$ and so we conclude that

$$\ddot{s} = -p_{\gamma}^{-1}[2\phi''(V - V_0) + p_{\gamma}(V - V_0)\dot{V}].$$

Now from (4) and (5), $\ddot{s} < 0$ so (8) is proved.

Weyl's statement now takes the form:

(9) Theorem. If $(V, p) \neq (V_0, p_0)$ is a point in $R$ at which $H=0$ and $dH \neq 0$, then $s|_{H=0}$ is not critical at $(V, p)$.

Proof. Since $dH \neq 0$ at $(V, p)$, the set $\{H=0\}$ meets a neighborhood of $(V, p)$ in a curve. Suppose the restriction of $s$ to this curve is critical at $(V, p)$; then, since $dH = \omega + Tds$, the restriction of $s$ to the integral curve of (6) through $(V, p)$ is also critical at $(V, p)$. Consider the negative half orbit, say $\gamma$, from $(V_0, p_0)$ to $(V, p)$ (definition of $R$): $H$ is zero at both ends of $\gamma$ so $H|_{\gamma}$ is critical at some point of $\gamma$ strictly between $(V_0, p_0)$ and $(V, p)$. Since $H$ and $s$ are critical together on $\gamma$ in (7) we find that $s$ has two critical points on an orbit of (6). This contradicts (8).

In order not to spoil the proof, we have not investigated the question of existence of points in $\{H=0\}$ at which $dH = 0$. In the case of a perfect gas, this does not happen.

REFERENCES


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