TOPOLOGICAL ALGEBRAS WITH A GIVEN DUAL
AJIT KAUR CHILANA

Abstract. Given an algebra $E$ and a total subspace $E'$ of its algebraic dual, we obtain necessary and sufficient conditions in terms of $E'$ for the existence of an $A$-convex or a locally $m$-convex topology on $E$ compatible with duality $(E, E')$. It has also been proved that if $E$ with the weak topology $\omega(E, E')$ is the closed linear hull of a bounded set and has hypocontinuous multiplication then it is locally $m$-convex.

1. Introduction. Let $E$ be a complex (or real) algebra and $E'$ be a total subspace of the algebraic dual $E^*$. To avoid repetitions we use the notation, terminology and results in [3] and [4] without specifications. An algebra with a locally convex linear topology for which multiplication is separately continuous will be called a locally convex algebra. An absolutely convex set $B$ in $E$ is called right (left) $A$-convex if it absorbs $Bx$ ($xB$) for each $x \in E$, it will be called $A$-convex if it is both right and left $A$-convex. A locally convex algebra is called (right, left) $A$-convex if there exists a basis of (right, left) $A$-convex neighbourhoods of zero. Multiplication in a locally convex algebra will be said to be right (left) hypocontinuous if given a neighbourhood $U$ of $0$ and a bounded set $B$ there exists a neighbourhood $V$ of $0$ satisfying $VB \subseteq U$ ($BV \subseteq U$). We say that multiplication is hypocontinuous if it is both right and left hypocontinuous. Gulick [5] has, however, called right hypocontinuity by hypocontinuity.

In §2 we answer the following question asked by Cochran [4].

(3.7) Under what conditions, in terms of $E'$, does $\Sigma(E, E')$ or $\chi(E, E')$—the finest $A$-convex or locally $m$-convex topology on $E$ compatible with duality $(E, E')$—exist?

It is known ([3] and [9], MR 41 #7435) that for $E$ with the weak topology $\omega(E, E')$ the conditions of joint continuity of multiplication, of $A$-convexity and of local $m$-convexity are mutually equivalent. We prove

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in §3 that if \((E, w(E, E'))\) is the closed linear hull of a bounded subset of itself then the condition of hypocontinuity of multiplication is also equivalent to all these conditions.

For \(y \in E\) and \(f \in E^*\), the right \(y\)-multiplicative translate \(f_y\) and the left \(y\)-multiplicative translate \(y_f\) of \(f\) are given by \(f_y(x) = f(xy)\) and \(y_f(x) = f(yx)\) for \(x \in E\) respectively. For \(y \in E\) and \(S \subseteq E^*\), let \(S(y) = \{f(y) : f \in S\}\), \(S_y = \{f_y : f \in S\}\) and \(yS = \{yf : f \in S\}\).

2. Topologies on \(E\) compatible with duality \((E, E')\).

(2.1) Definition. A set \(S \subseteq E^*\) is called collectionwise multiplicative if \(S(xy) \subseteq S(x)S(y)\) for all \(x, y \in E\).

(2.2) Definition. A set \(S \subseteq E^*\) is called collectionwise right (left) multiplicative-translation invariant if for each \(y \in E\) there is \(p_y^0 \geq 0\) satisfying \(S_y(x) \subseteq p_y^0 S(x)\) \((S(x) \subseteq p_y S(x))\) for all \(x \in E\). \(S\) will be called collectionwise multiplicative-translation invariant if it is both collectionwise right and collectionwise left multiplicative-translation invariant.

It is easy to see that every collection of multiplicative linear functionals is collectionwise multiplicative and every balanced, \(w(E^*, E)\)-bounded, collectionwise multiplicative subset of \(E^*\) is collectionwise multiplicative-translation invariant. Also an arbitrary union of collectionwise multiplicative sets is collectionwise multiplicative and a finite union of balanced collectionwise (right, left) multiplicative-translation invariant sets is collectionwise (right, left) multiplicative-translation invariant.

(2.3) Lemma. Let \(S \subseteq E'\) be balanced and \(w(E', E')\)-compact, and let \(S^o\) be its polar in \(E\).

(i) \(S^o\) is idempotent if and only if \(S\) is collectionwise multiplicative.

(ii) \(S^o\) is (right, left) \(A\)-convex if and only if \(S\) is collectionwise (right, left) multiplicative-translation invariant.

Proof. (i) Sufficiency is clear.

Necessity. For \(x \in E\), let \(p(x) = \sup \{|f(x)| : f \in S\}\). Since \(S\) is \(w(E', E)\)-compact, \(p(x) < \infty\) and there is an \(f \in S\) (depending on \(x\)) satisfying \(p(x) = |f(x)|\). Because \(S\) is balanced, \(g = \text{signum}(f(x)) \cdot f\) is in \(S\). So \(p(x) = g(x)\) for some \(g\) in \(S\). Also \(S^o = \{x \in E : p(x) \leq 1\}\) and \(p\) is its Minkowski functional. Now \(S^o\) is idempotent, so \(p\) is submultiplicative i.e. \(p(xy) \leq p(x)p(y)\) for all \(x, y \in E\).

Let \(x, y \in E\) and \(f \in S\). Then \(|f(xy)| \leq p(x)p(y)\). So there is a scalar \(\lambda\) such that \(|\lambda| \leq 1\) and \(f(xy) = \lambda p(x)p(y)\). Also there exist \(g\) and \(h\) in \(S\) (depending on \(x\) and \(y\) respectively) satisfying \(p(x) = g(x)\) and \(p(y) = h(y)\). If \(g_1 = \lambda g\) then \(g_1 \in S\). Thus \(f(xy) = g_1(x)h(y) \in S(x)S(y)\). Hence \(S(xy) \subseteq S(x)S(y)\) for all \(x, y \in E\) and \(S\) is collectionwise multiplicative.

(ii) Sufficiency is clear.
Suppose $S^\circ$ is right $A$-convex. For $y \in E$ there is $\lambda_y > 0$ such that $S^\circ y \leq \lambda_y S^\circ$. If $p$ is as in the proof of (i) above then $p$ satisfies all other properties except that submultiplicativity is replaced by $p(xy) \leq \lambda_y p(x)$ for all $x, y \in E$. So $|f(xy)| \leq p(xy) \leq \lambda_y p(x)$. Therefore, $f(xy) = \mu \lambda_y p(x)$ for some $\mu$ with $|\mu| \leq 1$. Let $g_\nu = \mu g$, where $g \in S$ is such that $p(x) = g(x)$. Then $f(xy) = g_\nu g(x)$. So $S(xy) \subseteq \lambda_y S(x)$ for all $x, y \in E$. Hence $S$ is collectionwise right multiplicative-translation invariant. Similarly we can prove for other parts.

(2.4) Theorem. There exists a locally $m$-convex topology on $E$ compatible with duality $(E, E')$ if and only if there exists a family $\mathcal{S}$ of absolutely convex, $w(E', E)$-compact, collectionwise multiplicative sets in $E'$ that cover $E'$.

(2.5) Corollary. The Mackey topology $\tau(E, E') = \chi(E, E')$ if and only if every absolutely convex, $w(E', E)$-compact set is contained in some absolutely convex, $w(E', E)$-compact, collectionwise multiplicative set in $E'$.

(2.6) Theorem. There exists a (right, left) $A$-convex topology on $E$ compatible with duality $(E, E')$ if and only if there is a family $\mathcal{S}$ of absolutely convex, $w(E', E)$-compact, collectionwise (right, left) multiplicative-translation invariant sets in $E'$ that cover $E'$.

(2.7) Corollary. $\tau(E, E') = \Sigma(E, E')$ if and only if every absolutely convex, $w(E', E)$-compact subset of $E'$ is contained in some absolutely convex, $w(E', E)$-compact, collectionwise multiplicative-translation invariant set.

(2.8) Remark. Since the existence of $\chi(E, E') (\Sigma(E, E'))$ is equivalent to the existence of some locally $m$-convex ($A$-convex) topology on $E$ compatible with $(E, E')$, Theorems (2.4) and (2.6) give an answer to question (3.7) in [4].

(2.9) Remark. If there are both $A$-convex and locally $m$-convex topologies on $E$ compatible with $(E, E')$ then $\chi(E, E') = \Sigma(E, E')$ if and only if every absolutely convex, $w(E', E)$-compact, collectionwise multiplicative-translation invariant set in $E'$ is contained in an absolutely convex, $w(E', E)$-compact, collectionwise multiplicative set in $E'$. This gives a partial answer to problem (3.6) in [4].

(2.10) Example. Let $E$ be the algebra of complex (or real) polynomials without constant term and $E'$ be the subspace of $E^*$ generated by $\{g_i: i=1, 2, \ldots\}$, where $g_i(e_j) = \delta_{ij}$, $e_j(x) = x^i$ for $i, j=1, 2, \ldots$. Then $(E, w(E, E'))$ is a locally $m$-convex algebra having no nonzero continuous multiplicative linear functionals (see Proposition 3 and discussion thereafter in [8]). By Theorem (2.4) there is a family $\mathcal{S}$ of absolutely convex,
w(E', E)-compact, collectionwise multiplicative sets in E' that cover E'. In fact, if \( G_n = \{ ng_i : 1 \leq i \leq n \} \), then its absolutely convex, \( w(E', E) \)-closed hull \( H_n \) in \( E' \) is \( w(E', E) \)-compact. Also the polar \( G_n^\circ \) of \( G_n \) in \( E \) is idempotent and \( H_n^\circ = G_n^\circ \). So by Lemma (2.3), \( H_n \) is collectionwise multiplicative.

This example shows that a collectionwise multiplicative set need not contain even a single nonzero multiplicative linear functional.

(2.11) Example. Let \( E \) be the algebra \( m \) of bounded complex (or real) sequences with pointwise addition and multiplication and let \( E' \) be the space \( l_1 \) of absolutely summable sequences. Then the Mackey topology \( \tau(E, E') \) is the same as the strict topology \( \beta \) on \( E \) considered as the space \( C_\varnothing(S) \) of bounded continuous complex (or real) functions on the space \( S \) of positive integers with the discrete topology ([2], [3], and [4]). Let \( \kappa \) be the compact open topology on \( E \). By Corollary (3.3) in [4], there is no locally \( m \)-convex topology on \( E \) between \( \beta \) and \( \kappa \). The dual of \((E, \kappa)\) is the space of sequences with only a finite number of nonzero elements and therefore \( \kappa < w(E, E') \).

(i) \( E \) is not locally \( m \)-convex under any topology compatible with \((E, E')\). So there exists no family of absolutely convex, \( w(l_1, m) \)-compact (and therefore, \( \| \cdot \|_1 \)-compact), collectionwise multiplicative sets that cover \( l_1 \).

(ii) \((E, \beta)\) has the Mackey topology and is \( A \)-convex [4]. So every absolutely convex, \( w(l_1, m) \)-compact subset of \( l_1 \) is contained in an absolutely convex, \( w(l_1, m) \)-compact, collectionwise multiplicative-translation invariant set.

3. \( E \) with the weak topology \( w(E, E') \). In this section \( E \) will denote the space \( E \) with the weak topology \( w(E, E') \). For \( B \subset E \) let \( E_B \) denote the linear hull of \( B \).

(3.1) Lemma. Suppose that \( E \) has hypocontinuous multiplication. Let \( g \) be in \( E' \) and \( B \) be an absolutely convex bounded subset of \( E \). Then the kernel \( K(g) \) of \( g \) contains a closed subspace \( J \) of finite codimension in \( E \) such that \( K(g) \) contains \( JE_B \) and \( E_B J \).

Proof. Let \( V \) be the polar of \( \{ g \} \) in \( E \). Since the multiplication in \( E \) is hypocontinuous there exists a finite set \( F = \{ f_i : 1 \leq i \leq n \} \) such that \( V \supset (BF^\circ) \cup (F^\circ B) \). Let \( J = \{ x \in E : f_i(x) = 0, \ 1 \leq i \leq n \} \). Then \( JB \subset F^\circ B \subset V \) and also \( J \) is a closed subspace of finite codimension in \( E \). Also \( JE_B = JB \subset V = \{ g \}^\circ \) and as \( JE_B \) is a linear space \( JE_B \subset K(g) \). Similarly \( E_B J \subset K(g) \).

(3.2) Theorem. If \( E \) is the closed linear hull of a bounded subset of itself and \( E \) has hypocontinuous multiplication then \( E \) has jointly continuous multiplication.
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Proof. Let \( B \) be an absolutely convex bounded subset of \( E \) such that \( E = E_B^- \), where '−' denotes the closure in \( E \). Let \( g \) be in \( E' \). Let \( J \) be as in the proof of the above lemma. Then \( J E = (J E_B)^- \subseteq (J E_B)^- \subseteq (K(g))^- = K(g) \). Similarly, \( EJ \subseteq K(g) \). Theorem 2 of Warner \[8\] now gives that \( E \) has jointly continuous multiplication.

(3.3) Corollary. If \( E \) is the closed linear hull of a bounded set then \( E \) is locally \( m \)-convex if and only if \( E \) is \( A \)-convex if and only if it has jointly continuous multiplication if and only if it has hypocontinuous multiplication.

Proof. Combine Theorem (3.4) in \[3\], Theorem 1 in \[9\] and Theorem (3.2) above.

(3.4) Remark. If a locally convex Hausdorff space is the closed linear hull of a bounded set i.e. it is \textit{boundedly generated} (in short, BG) in the terminology of \[6\] then it is BG under each topology compatible with duality (Remark 10 in \[1\]). Every normed linear space is BG and a product of BG spaces is again BG \( \{6\} \) (see also Remark 10 in \[1\] and \[2\]). Thus our results are applicable to a large class of algebras.

(3.5) Example. The algebra \((m, w(m, t))\) is BG but not locally \( m \)-convex \((\{2\}, \) and Example (2.11) (i) above). So it is not \( A \)-convex and does not have hypocontinuous multiplication.

(3.6) Example. Let \( E \) be the algebra of all complex (or real) continuous functions on the interval \([0, 1]\) with pointwise addition and multiplication equipped with the weak topology resulting from the sup norm topology. Then \( E \) is a BG space. Warner \[8\] has shown that \( E \) does not have jointly continuous multiplication. Therefore, \( E \) is not \( A \)-convex and \( E \) does not have hypocontinuous multiplication. Thus the claim made in the second part of Examples 3.12 in \[5\] is not valid.

(3.7) Example. Consider the algebra \( \varphi \) of complex (or real) sequences with only a finite number of nonzero elements. Then its algebraic dual is the space \( \omega \) of all complex (or real) sequences under the duality given by \( f(x) = \sum_{n=1}^{\infty} \xi_n \xi_n \) for \( x = (\xi_n) \in \varphi \) and \( f = (\zeta_n) \in \omega \). So the Mackey topology \( \tau(\varphi, \omega) \) is the finest locally convex topology on \( \varphi \) and therefore is the same as the direct sum topology. Also bounded sets are finite-dimensional and every absolutely convex absorbent set is a neighborhood of \( o \) in \( \varphi \). Moreover, \( \omega \) is the \( \alpha \)-dual of \( \varphi \) and \( \tau(\varphi, \omega) \) is the same as the normal topology, a base of neighborhoods of \( o \) which is given by

\[
U_f = \left\{ x = (\xi_n) \in \varphi : \sum_{n=1}^{\infty} |\xi_n \xi_n| \leq 1 \right\}, f = (\zeta_n) \in \omega \quad [7, \S30.1].
\]

Let \( V_f = \{ x \in \varphi : \sum_{n=1}^{\infty} |\xi_n \xi_n| \leq 1, \sum_{n=1}^{\infty} |\xi_n \eta_n \xi_n| \leq \sum_{n=1}^{\infty} |\eta_n \xi_n| \text{ for all } y = (\eta_n) \in \varphi \}. \) Then \( V_f, V_f \subseteq V_f \subseteq U_f \) and also \( V_f \) is an absolutely convex
absorvent set and thus a neighbourhood of $o$ in $\tau(\varphi, \omega)$. So $\tau(\varphi, \omega)$ is locally $m$-convex.

Now let $E$ denote the space $\varphi$ with the weak topology $w(\varphi, \omega)$. Then $E$ has hypococontinuous multiplication but does not have jointly continuous multiplication.

If $B$ is bounded on $E$ then there exists an integer $N$ and an $\alpha \geq 0$ such that $B \subseteq \{x=(\xi_n): \xi_n=0 \text{ for } n>N \text{ and } |\xi_n| \leq \alpha \text{ for } n \leq N\}$. Let $f=(\xi_n) \in E' = \omega$ and let $U$ be its polar in $E$. For $n \leq N$, let $g_n \in E'$ be given by $g_n(x) = N\alpha |\xi_n| \xi_n$, $x=(\xi_n) \in E$. Then the polar $V$ of $\{g_n: 1 \leq n \leq N\}$ is a neighbourhood of $o$ in $E$. Also $VB \subseteq U$. Thus $E$ has hypococontinuous multiplication.

Now consider $f=(\xi_n) \in E'$ given by $\xi_n=1$ for all $n$. If $E$ is locally $m$-convex then by Theorem 1 of [8], the kernel $K(f)$ of $f$ contains an ideal $J$ of finite codimension. Let $x \neq 0 \in J$. Let $y=(\eta_n) \in E$ be given by $\eta_n = \xi_n (n=1, 2, \cdots)$. Then $xy \in J$. Now $f(xy) = \sum_{n=1}^{\infty} |\xi_n|^2 \neq 0$. So $xy \notin K(f)$, which gives a contradiction. So $E$ is not locally $m$-convex and is, therefore, not $A$-convex and does not have jointly continuous multiplication.

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