PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 42, Number 1, January 1974

THE DENSITY CHARACTER OF FUNCTION SPACES

N. NOBLE

ABSTRACT. For topologies between the pointwise topology and the compact-open topology, the density character of $C(X)$ (and, for certain spaces Z, $C(X, Z)$) is described in terms of a cardinal invariant of X. The Hewitt-Pondiczery theorem on the density character of product spaces follows as a corollary.

1. **Description.** Except in Corollary 2, all hypothesized spaces are assumed to be completely regular Hausdorff. The set of continuous functions from X to Z is denoted by $C(X, Z)$ or, when $Z=R$, by $C(X)$. The *density character*, δX, of a space X is the least cardinality of a dense subset of X and the *weight*, wX, of X is the least cardinality of an open basis of X. We define the *weak weight*, wwX, of X to be the least of the cardinals wY for Y a continuous one-to-one image of X.

THEOREM 1. *Let X be any infinite space and let $C(X)$ have any topology between the pointwise topology and the compact-open topology. Then $\delta C(X)=wwX$.*

All proofs will be given in the next section. Our remaining results allow Theorem 1 to be applied to yield information about $\delta C(X, Z)$ for suitable paces Z.

LEMMA. *Let $C(X, Z)$ and $C(X, Z^*)$ both have either the topology of uniform convergence on members of some fixed cover of X or the set-open topology generated by such a cover. If Z is a retract of Z^* then $\delta C(X, Z) \leqq \delta C(X, Z^*)$.*

PROPOSITION. *For any topologies between the pointwise topology and the compact-open topology:*

(i) *For any space Z, $\delta Z \leqq \delta C(X, Z)$.*

(ii) *If Z contains a nondegenerate arc, then $\delta C(X) \leqq \delta C(X, Z)$.*

(iii) *If for each finite subset F of X there exists a function f in $C(X, Z)$ such that F and $f(F)$ have the same cardinality, then $\delta C(X) \leqq \delta C(X, Z) \cdot \delta C(Z)$.*

Received by the editors February 26, 1971.

AMS (MOS) subject classifications (1970). Primary 46E10, 54C35; Secondary 54D10.

Key words and phrases. Density character, separable spaces, function spaces.

THEOREM 2. *Suppose Z contains more than one point and is a retract of a convex subset of some locally convex linear space. For any topologies between the pointwise topology and the compact-open topology, $\delta C(X, Z) = \delta C(X) \cdot \delta Z$.*

COROLLARY 1. *For Z as in Theorem 2 and any topology between the pointwise topology and the compact-open topology, $C(X, Z)$ is separable if and only if Z is separable and some continuous one-to-one image of X is separable metrizable.*

Under the same hypotheses on a separable space Z, Vidossich [6] shows that $C(X, Z)$ is separable if and only if X is submetrizable (has a continuous one-to-one metrizable image) with $\delta X \leqq c$, which is equivalent to being submetrizable and of cardinality less than or equal to c. Combined with Corollary 1 this yields:

COROLLARY 2. *On a set of cardinality less than or equal to c each metrizable topology contains a separable metrizable topology.*

THEOREM 3. *For the topology of pointwise convergence, if Z is arcwise connected or if X is a P-space then $\delta C(X, Z) \leqq \delta C(X) \cdot \delta Z$.*

COROLLARY 3 (HEWITT [2], PONDICZERY [3]). *Let $\mathfrak{n}$ be an infinite cardinal. With no separation axioms assumed, let $X = \prod_{\alpha \in A} X_\alpha$ where each X_α has $\delta X_\alpha \leqq \mathfrak{n}$. If* card $A \leqq 2^{\mathfrak{n}}$, *then $\delta X \leqq \mathfrak{n}$.*

Turning to other topologies on $C(X)$ we note that the compact-open topology arises in Theorem 1 only because we need the Stone-Weierstrass Theorem (that a subalgebra is dense if it contains the constants and separates points). When dealing with a subalgebra generated by a set of cardinality $\mathfrak{n}$ which separates points, compactness can be weakened to weak-$\mathfrak{n}$-$\aleph_0$-compactness (each $\mathfrak{n}$-fold open cover has a finite subcollection whose union is dense—such spaces are studied, for example, in [4]) so if $wwX = \mathfrak{n}$, then in the set-open topology generated by all weakly-$\mathfrak{n}$-$\aleph_0$-compact subsets of X, $\delta C(X) \leqq \mathfrak{n}$. In case $wwX = \aleph_0$, an even stronger statement holds: $C(X)$ is separable in the set-open topology generated by all relatively pseudocompact subsets of X. (A set $S \subseteq X$ is *relatively pseudocompact* if each function in $C(X)$ is bounded on S; weak-$\aleph_0$-$\aleph_0$-compactness is equivalent to pseudocompactness.)

Of course the most interesting topology on $C(X)$ larger than the compact-open topology is the topology of uniform convergence, and for this topology it is known that $\delta C(X) = w\beta X$: That $\delta C^*(X) = w\beta X$ is established by Smirnov in [5] by a proof similar to the proof of Theorem 1 and the more difficult result $\delta C^*(X) = \delta C(X)$ is shown by Comfort and Hager in [1].

Since the largest set-open topology on $C(X)$ is the topology of uniform convergence if (and only if) X is pseudocompact, it follows by the Comfort-Hager-Smirnov Theorem, the comments above, and the fact that βX is separable only if X is compact, that if X is compact separable metric, no strictly larger topology on X is pseudocompact.

2. **The proofs.**

PROOF OF THEOREM 1. We first show that $\delta C(X) \leqq wwX$; since larger topologies have "larger" density characters, it suffices to consider the case where $C(X)$ has the compact-open topology. Let $\mathscr{B}$ be a base for some topology contained in the topology of X with card $(\mathscr{B}) = wwX$ and form a set of functions $D \subseteq C(X, I)$ by choosing, for each pair (U, V) in $\mathscr{B} \times \mathscr{B}$ with $U \subseteq V$, a function which maps U to 0 and $X \setminus V$ to 1, if such a function exists. Clearly D separates points of X, so by the Stone-Weierstrass Theorem (for example, [7, problem 44B]) the algebra generated by D and the constant functions is dense in $C(X)$. Hence the set, D^*, of all finite linear combinations, with rational coefficients, of members of D or rational constant functions is dense in $C(X)$. Since card $D^* =$ card $D = wwX$, this shows that $\delta C(X) \leqq wwX$.

For the remaining relation, $wwX \leqq \delta C(X)$, it suffices to consider $C(X)$ in the pointwise topology. Let $D \subseteq C(X)$ be dense with card $(D) = \delta C(X)$ and let $e: X \to R^D$ be the embedding map, $e(x)_d = d(x)$. Since D is dense, it separates points, so e is one-to-one. Since each subspace of R^D has weight $\leqq$ card $D = \delta C(X)$, it follows that $wwX \leqq \delta C(X)$.

PROOF OF THE LEMMA. For $D \subseteq C(X, Z^*)$ dense and $r: Z^* \to Z$ a retraction, $\{r \circ d : d \in D\}$ is dense in $C(X, Z)$.

PROOF OF THE PROPOSITION. (i) Identifying Z with the constant functions and choosing z in Z, the function which carries f to $f(z)$ is a retraction of $C(X, Z)$ to Z. Thus, by the lemma, $\delta Z \leqq \delta C(X, Z)$.

(ii) By the lemma $\delta C(X, I) \leqq \delta C(X, Z)$ and since

$$C^*(X) = \bigcup_n C(X, [-n, n])$$

is dense in $C(X)$, $\delta C(X) = \delta C(X, I)$. Therefore $\delta C(X) \leqq \delta C(X, Z)$.

(iii) To see that $\delta C(X) \leqq \delta C(X, Z) \cdot \delta C(Z)$ it suffices to consider the pointwise topologies, since by Theorem 1, $\delta C(X)$ does not change, and $\delta C(X, Z)$ can only be larger. Let D_1 and D_2 be dense in $C(X, Z)$ and $C(Z)$ respectively with card $D_1 = \delta C(X, Z)$, card $D_2 = \delta C(Z)$, and set $D = \{g \circ f : f \in D_1, g \in D_2\}$. We show that D is dense in $C(X)$. Given distinct points $x_1, \cdots, x_n$ in X and open subsets $U_1, \cdots, U_n$ of R with $\bigcap_{i=1}^n N(x_i, U_i) \neq \varnothing$ (where $N(x, U) = \{f : f(x) \in U\}$), there exists a function h in $C(X, Z)$ such that the points $h(x_i)$ are all distinct. Choosing disjoint open neighborhoods V_i of $h(x_i)$, $\bigcap_{i=1}^n N(x_i, V_i)$ is thus a nonempty open

set, so it contains a function f in D_1. Since $\bigcap_{i=1}^n N(x_i, U_i) \neq \varnothing$, it contains a function which maps the points x_i to distinct reals and hence, shrinking the U_i if necessary, we may suppose that they are all disjoint. Thus $\bigcap_{i=1}^n N(f(x_i), U_i)$ is not empty, so it contains a function g in D_2. But now $g \circ f$ is in $\bigcap_{i=1}^n N(x_i, U_i)$, as desired.

PROOF OF THEOREM 2. Since Z must contain a nondegenerate arc, the relation $\delta C(X, Z) \geqq \delta C(X) \circ \delta Z$ follows from the first two parts of the proposition. For the reverse inequality it suffices to consider the case where Z is a convex subset of a locally convex linear space Z^* which contains the origin and where $C(X, Z)$ and $C(X)$ bear the compact-open topologies. Let D' be a dense subset of $C(X, I)$ of cardinality $\delta C(X)$ (that $\delta C(X, I) = \delta C(X)$ follows from the lemma and the second part of the proposition) and let D'' be the set of convex combinations, with rational coefficients, of members of D'. For g and g' in D'' and for r and s rational with $0 \leqq r < s \leqq 1$ choose $h(g, g', r, s)$ in $C(X, I)$ such that for $h = h(g, g', r, s)$, $h(x) \leqq g'(x)$ for all x in X, $h(x) = g'(x)$ if x is in $g^{-1}([0, r])$, and $h(x) = 0$ if x is in $g^{-1}([s, 1])$. Finally let D_1 be the set of all functions $h(g, g', r, s)$ so chosen and note that the cardinality of D_1 is $\delta C(X)$. Now let D_2 be a dense subset of Z of cardinality δZ and, identifying z in D_2 with the constant function on X whose value is z, set

$$D = \left\{ \sum_{i=1}^m f_i z_i : f_i \in D_1,\ z_i \in D_2 \right\}.$$

Since the cardinality of D is $\delta C(X) \cdot \delta Z$, it suffices to show that $D \cap C(X, Z)$ is dense in $C(X, Z)$.

Let K be a compact subset of X, let ρ be a continuous seminorm for Z^*, let ε be a positive rational less than 1 and choose f in $C(X, Z)$. Let $S = \{z \in Z : \rho(z) < \varepsilon/2\}$, cover K with the sets $U(x) = f^{-1}(f(x) + S)$ for $f(x)$ in D_2 and let $U(x_1), \cdots, U(x_n)$ be a finite subcover. Let $\Phi_1, \cdots, \Phi_n$ be a partition of unity subordinate to this cover, let M be a positive integer greater than each of the numbers $2n\rho(f(x_i))$, $1 \leqq i \leqq n$, and choose functions f_i in D' such that for each x in K, $\Phi_i(x) - \varepsilon/M < f_i(x) < \Phi_i(x) - \varepsilon/2M$. Set $g = (1/n) \sum_{i=1}^n f_i$, $r = 1/n - \varepsilon/2M$ and $s = 1/n$ and let $h_i = h(g, f_i, r, s)$, $1 \leqq i \leqq n$. Note that for x in K,

$$g(x) = \left(\frac{1}{n}\right)\left(\sum_{i=1}^n f_i(x)\right) \leqq \left(\frac{1}{n}\right)\left(\sum_{i=1}^n \Phi_i(x) - \frac{n\varepsilon}{2M}\right) = r,$$

so $h_i(x) = f_i(x)$. Furthermore, for any x, $\sum_{i=1}^n h_i(x) \leqq \sum_{i=1}^n f_i(x)$ and $\sum_{i=1}^n h_i(x) = 0$ if $\sum_{i=1}^n f_i(x) \geqq 1$, so $\sum_{i=1}^n h_i(x) \leqq 1$. Therefore the function $h = \sum_{i=1}^n h_i f(x_i)$ is in $D \cap C(X, Z)$. We complete the proof by showing that

$\rho(h(x)-f(x))<\varepsilon$ for x in K. For such x,

$$\begin{aligned}\rho(h(x)-f(x)) &= \rho\left(\sum_{i=1}^{n} h_i(x)f(x_i) - \sum_{i=1}^{n}\Phi_i(x)f(x)\right)\\ &= \rho\left(\sum_{i=1}^{n}(h_i(x)-\Phi_i(x))f(x_i) + \sum_{i=1}^{n}\Phi_i(x)(f(x_i)-f(x))\right)\\ &\leqq \sum_{i=1}^{n}\rho(f(x_i))\,|\Phi_i(x)-f_i(x)| + \left(\sum_{i=1}^{n}\Phi_i(x)\right)(\varepsilon/2)\\ &\leqq \varepsilon/2+\varepsilon/2=\varepsilon.\end{aligned}$$

PROOF OF THEOREM 3. We first suppose that Z is arcwise connected. Let D' be a dense subset of Z of cardinality δZ and for each finite subset F of D' choose an arcwise connected subspace X_F of Z which contains F, is the union of finitely many arcs, and which contains no closed loops. (To construct such a subspace, suppose inductively that Y_k has been constructed so as to satisfy these conditions and contain k points of F, let z be a point in $F\setminus Y_k$, let f be an arc with $f(0)$ in Y and $f(1)=z$ and let r be the largest number such that $f(r)$ is in Y_k. The space $Y_{k+1}=Y_k\cup f([r,1])$ has the required properties and contains $k+1$ points of F.) The space X_F is a retract of R^2, so $\delta C(X, X_F)\subseteq\delta C(X)$ and hence $C(X, X_F)$ contains a dense subset D_F of cardinality at most $\delta C(X)$. Set $D=\bigcup\{D_F: F \text{ is a finite subset of } D'\}$ and note that the cardinality of D is at most $\delta C(X)\cdot\delta Z$.

Observe that each X_F can be expressed as $h_F(I)$ for some continuous function h_F. Hence if $F'\subseteq X$ is finite and $f:F'\to F\subseteq D'$ is given, then f has a continuous extension $f^*:X\to X_F$. (Indeed, for $g:F'\to I$ with each $g(x)$ in $h_F^{-1}(f(x))$ and $g^*:X\to I$ a continuous extension of g, the function $f^*=h_F\circ g^*$ is such an extension.) It is now easy to show that D is dense. Given distinct points $x_1,\cdots,x_n$ of X and nonempty open subsets $U_1,\cdots,U_n$ of Z, choose points z_i in $D'\cap U_i$, set $F=\{z_1,\cdots,z_n\}$, let f be the function which carries x_i to z_i and let $f^*:X\to X_F$ be a continuous extension of f. Then f^* is in $\bigcap_{i=1}^n N(x_i, U_i)$, so $\bigcap_{i=1}^n N(x_i, U_i)$ is not empty. It follows that D_F, and hence D, meets $\bigcap_{i=1}^n N(x_i, U_i)$, as desired.

Now suppose that X is a P-space. Let Z^* be the quotient of $I\times Z$ formed by identifying $\{1\}\times Z$ to a point v, choose a point z_0 in Z and let $q:Z^*\to Z$ be the map which carries v to z_0 and points (r, z) to z. Since Z^* is arcwise connected, $C(X, Z^*)$ contains a dense subset D' of cardinality at most $\delta C(X)\cdot\delta Z$. Note that if $f:X\to Z^*$ is continuous then $q\circ f$ is continuous since $f^{-1}(v)$, being a closed G_δ, is both closed and open. Thus $D=\{q\circ f: f\in D'\}$ is a subset of $C(X, Z)$ of cardinality at most $\delta C(X)\cdot\delta Z$. Since D is clearly dense in the pointwise topology, the proof is complete.

PROOF OF COROLLARY 3. Let $X=\prod_{\alpha\in A} X_\alpha$ where card $A=2^{\mathfrak{n}}$ and where, we may suppose, A is discrete and each X_α contains a dense subset D_α of cardinality $\mathfrak{n}$. Let $Z_\alpha=Z$, the discrete space of cardinality $\mathfrak{n}$, and let D' be a dense subset of $\prod_{\alpha\in A} Z_\alpha=C(A,Z)$ of cardinality at most $\delta C(A)\cdot\delta Z=wwA\cdot\delta Z=\mathfrak{n}\cdot\mathfrak{n}=\mathfrak{n}$. (Since A is discrete, wwA is the least cardinal $\mathfrak{m}$ such that card $A\leqq$card $R^{\mathfrak{m}}=2^{\mathfrak{m}}$, so $wwA=\mathfrak{n}$.) Let $i_\alpha:Z_\alpha\to D_\alpha$ be one-to-one onto and set $D=(\prod_\alpha i_\alpha)(D')$. Since the projection of D' onto any finite subproduct must be onto, it is clear that D is dense, as desired.

REFERENCES

1. W. Comfort and A. Hager, *Dense subspaces of some spaces of continuous functions*, Math. Z. **114** (1970), 373–389.

2. E. Hewitt, *A remark on density characters*, Bull. Amer. Math. Soc. **52** (1946), 641–643. MR **8**, 139.

3. E. S. Pondiczery, *Power problems in abstract spaces*, Duke Math. J. **11** (1944), 835–837. MR **6**, 119.

4. V. Saks and R. M. Stephenson, Jr., *Products of $\mathfrak{M}$-compact spaces*, Proc. Amer. Math. Soc. **28** (1971), 279–288. MR **42** #8448.

5. Yu. M. Smirnov, *On the weight of the ring of bounded continuous functions over a normal space*, Mat. Sb. **30(72)** (1952), 213–218. (Russian) MR **14**, 70.

6. G. Vidossich, *Characterizing separability of function spaces*, Invent. Math. **10** (1970), 205–208. MR **42** #2424.

7. S. Willard, *General topology*, Addison-Wesley, Reading, Mass., 1970. MR **41** #9173.

CANARY ROAD, WESTLAKE, OREGON 97493