A NOTE ON GROUPS WITH RELATIVELY COMPACT CONJUGACY CLASSES

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Abstract. In a more general form, the following structure theorem is proved. Let $G$ be a locally compact group with small invariant neighborhoods. Then $G$ has relatively compact conjugacy classes if and only if $G$ is a direct product of a vector group $V$ and a group $L$ where $L$ has a compact open normal subgroup $K$ such that $L/K$ has finite conjugacy classes.

The purpose of this note is to prove the following theorem which is a direct generalization of the basic structure theorem for locally compact abelian groups [2, Theorem 24.30].

Theorem. Let $\mathcal{B}$ be a subgroup of $\mathcal{A}(G)$ containing the inner automorphisms. Let $G \in [SIN]_\mathcal{B}$. Then $G \in [FC]_\mathcal{B}$ if and only if $G$ contains $\mathcal{B}$-invariant subgroups $V$, $L$ and $K$ such that $V$ is a vector group, $K$ is compact and open in $L$, $L/K \in [FC]_\mathcal{B}$, and $G = VL$ is a direct product of $V$ and $L$.

First we establish a few definitions and some notation. All groups considered are Hausdorff and locally compact. The group operation is multiplication. A vector group is one which is topologically isomorphic to the additive structure of $\mathbb{R}^n$ with $n \geq 0$. The connected component of the identity of a topological group $G$ is denoted $G_e$. An element of $G$ is said to be compact if the subgroup it generates has compact closure. The group of topological automorphisms of $G$ is $\mathcal{A}(G)$. If $\mathcal{B}$ is a subgroup of $\mathcal{A}(G)$ which contains the inner automorphisms, then the $\mathcal{B}$-orbit of $x \in G$ is $\{\beta(x) : \beta \in \mathcal{B}\}$. A subset $S$ of $G$ is said to be $\mathcal{B}$-invariant, if $\beta(s) \in S$ for all $s \in S$ and $\beta \in \mathcal{B}$. We are interested in the following classes of groups.

$G \in [FC]_\mathcal{B}$ if the $\mathcal{B}$-orbits of points have compact closures.

$G \in [SIN]_\mathcal{B}$ if there is a neighborhood basis of compact $\mathcal{B}$-invariant neighborhoods at the identity.

$G \in [FD]_\mathcal{B}$ if the $\mathcal{B}$-commutator subgroup, which is the closure of the group generated by $\{x^{-1}\beta(x) : \beta \in \mathcal{B}, x \in G\}$, is compact.

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If \( \mathcal{B} \) actually equals the inner automorphism group, then the subscript and prefix \( \mathcal{B} \) is omitted. If \( H \) is a \( \mathcal{B} \)-invariant subgroup of \( G \), then the restriction of \( \mathcal{B} \) to \( H \) is a subgroup of \( \mathfrak{A}(H) \) which, by abuse of notation, is again denoted \( \mathcal{B} \). Similar remarks apply to quotients formed by \( \mathcal{B} \)-invariant subgroups.

The proof of the theorem relies on the following results of Grosser and Moskowitz and on the lemma below. Let \( \mathcal{B} \) be a subgroup of \( \mathfrak{A}(G) \) containing the inner automorphisms; let \( G \in [ FC ]_{\mathcal{B}} \) and let \( P \) be the periodic subgroup of \( G \), that is, \( P \) is the set of compact elements of \( G \).

1. \( P \) is a closed \( \mathcal{B} \)-invariant subgroup of \( G \) and the sequence \( 1 \to P \to G \to W \times D \to 1 \) is exact. Here \( W \) is a vector group and \( D \) is a discrete torsion-free abelian group [3, Theorem 3.16].

2. If \( G \) is compactly generated, then \( P \) is compact [3, Theorem 3.20].

3. If \( G \in [ FD ]^{-} \), then normal vector subgroups split [3, Corollary 4.3].

**Lemma.** Let \( \mathcal{B} \) be a subgroup of \( \mathfrak{A}(G) \) containing the inner automorphisms and let \( G \in [ FC ]_{\mathcal{B}} \). If the connected component of the identity \( G_{e}=VK \) is a direct product of a nontrivial \( \mathcal{B} \)-invariant vector subgroup \( V \) and a compact group \( K \), then there is a \( \mathcal{B} \)-invariant subgroup \( L \) such that \( G=VL \) is a direct product of \( V \) and \( L \) with \( L_{e} \) compact.

**Proof.** Let \( P \) be the set of compact elements of \( G \). We claim the map \( \psi: VP/P \to V/(V \cap P)=V \) defined by \( \psi(vP)=v \) is a topological isomorphism. Since \( V \) is \( \sigma \)-compact and \( P \) is closed, this follows from [2, Theorem 5.33] providing we show that \( VP \) is open, hence closed in \( G \). Let \( H \) be any compactly generated open subgroup of \( G \). Then \( H \in [ FC ]^{-} \) and so by (2) \( H \in [ FD ]^{-} \). Furthermore, \( V \) is a normal vector subgroup of \( H \) so that, by (3), \( H=VM \) is a direct product of \( V \) with a subgroup \( M \). Since \( H=VM \supseteq G_{e}=VK \) and \( V \) contains no compact elements, \( M \supseteq K \). Furthermore, \( M/K \cong H/G_{e} \) is totally disconnected. Thus \( M \) contains a compact open subgroup \( M_{1} \). Since \( VM_{1} \subset VP \) and \( VM_{1} \) is an open subset of \( G \), \( VP \) is open in \( G \).

By (1) \( G/P=WD \) is a direct product of a vector subgroup \( W \) with a discrete subgroup \( D \). Let \( \pi_{1}:G\to G/P \) and \( \pi_{2}:WD\to W \) be the canonical projections. Next note that \( \pi_{1}^{-1}(\pi_{1}(G_{e}))=G_{e}P=VKP=VP \) is open implies that \( \pi_{1}(G_{e}) \) is open and hence closed and so \( \pi_{1}(G_{e})=VP/P=(G/P)_{e}=W \) [2, Theorem 7.12]. That is, \( W=VP/P \). Now consider the composition

\[
G \xrightarrow{\pi_{1}} G/P = (VP/P)D \xrightarrow{\pi_{2}} VP/P \xrightarrow{\psi} V.
\]

If \( v \in V \), then \( \psi(\pi_{2}(\pi_{1}(v)))=\psi(\pi_{2}(vP))=\psi(vP)=v \). Thus \( \pi=\psi \circ \pi_{2} \circ \pi_{1} \) is a projection onto the normal subgroup \( V \) and \( G=VL \) is a direct product with \( L=\ker \pi \).
We now show that $L$ is $\mathcal{B}$-invariant. Let $x \in L$ and let $O(x)$ be the closure of the $\mathcal{B}$-orbit of $x$ which is compact and $\mathcal{B}$-invariant. Let $G_x$ be the subgroup of $G$ generated by $O(x)$. Then $G_x$ is $\mathcal{B}$-invariant and so is a compactly generated $[FC]_{\mathcal{B}}$ group. By (2) $G_x \in [FD]_{\mathcal{B}}$. This means that the $\mathcal{B}$-commutator subgroup of $G_x$ is compact so that its image in $V$ under $\pi$ is a compact, hence trivial, subgroup of $V$. It follows that $x^{-1} \beta(x) \in \ker \pi = L$ and $\beta(x) \in xL = L$, for each $\beta \in \mathcal{B}$. Since $x$ was an arbitrary element of $L$, $L$ is $\mathcal{B}$-invariant. Since $L_e = (G/V)_e = G_e/V = K$, $L_e$ is compact [2, 7.13].

Proof of the Theorem. Assume $G \in [FC]_{\mathcal{B}} \cap [SIN]_{\mathcal{B}}$. Then

$$G_e \in [FC]_{\mathcal{B}} \cap [SIN]_{\mathcal{B}}$$

so that the closure of $\mathcal{B}$ as a subgroup of $\mathfrak{A}(G_e)$ is compact [3, Theorem 0.1]. Since $G_e$ is a connected $[SIN]$-group, it is maximally almost periodic and is a direct product $G_e = V_1 K_1$ of a vector group $V_1$ and a compact group $K_1$ [1, Théorème 16.4.6]. Since $K_1$ is a characteristic subgroup of $G_e$, there is an automorphism $\alpha$ of $G_e$ such that $V_1 = \alpha(V_1)$ is a $\mathcal{B}$-invariant subgroup of $G$ and $G_e = V K_1$ [3, Theorem 1.1]. The lemma now applies and we have $G = VL$ with the desired properties. All that remains is to exhibit the required compact open subgroup $K$ of $L$. The totally disconnected group $L/L_e$ is in $[SIN]_{\mathcal{B}}$ so that any compact open subgroup $K_2$ in $L/L_e$ contains a $\mathcal{B}$-invariant neighborhood of the identity. Thus $\bigcap \{ \beta K_2 : \beta \in \mathcal{B} \}$ is a compact open $\mathcal{B}$-invariant subgroup of $L/L_e$. Let $K$ be its inverse image in $L$.

Conversely, assume $G = VL$ as in the statement of the theorem. It suffices to show that $L \in [FC]_{\mathcal{B}}$. Let \{\$x\alpha K : \alpha \in A$\} be a coset decomposition of the discrete group $L/K$. Let $x \in L$ so that $x = x_\alpha k$ for some $\alpha$. The $\mathcal{B}$-orbit $O$ of $x_\alpha K$ is finite. Thus, if $\pi$ is the projection of $L$ on $L/K$, we have $\pi(\beta(x_\alpha)) \in O$. Consequently, $\beta(x) = \beta(x_\alpha) \beta(k) \in \pi^{-1}(O)K$. That is, the $\mathcal{B}$-orbit of $x$ is contained in a compact subset of $L$.

Remarks. The theorem stated above is a generalization of a structure theorem of Grosser and Moskowitz [3, Theorem 4.6]. In their case the group $G$ was in $[FD]_{\mathcal{B}}$ and they were able to choose the compact subgroup $K$ so that $L/K$ was $\mathcal{B}$-fixed. That this is not generally possible for $G \in [FC]_{\mathcal{B}}$ is illustrated by considering a group $G$ which is a discretely topologized weak direct sum of an infinite number of copies of a finite simple group [3, p. 39]. This group has finite conjugacy classes and the existence of such a (finite) subgroup $K$ would imply that $G$ had a finite commutator subgroup, which it does not.

Compactly generated locally compact abelian groups split as a direct product $R^n \times Z^m \times K$, with $K$ compact [2, Theorem 9.8]. This theorem does not generalize to any reasonable class of nonabelian groups. However,
if we assume that $G$ is a compactly generated group in $[SIN]_{\mathbb{g}}$, the theorem remains valid with \( L/K \cong \mathbb{Z}^m \) for some \( m \geq 0 \) and \( L/K \) is $\mathbb{g}$-fixed” replacing \( L/K \in [FC]_{\mathbb{g}} \). This can be obtained as a corollary of our theorem by utilizing the method of proof of [3, Proposition 4.5] as outlined below. Without loss of generality, we can now assume that $K$ contains the $\mathbb{g}$-commutator subgroup of $L$ so that $L/K$ is a finitely generated abelian group and then enlarge $K$ so that $L/K$ is torsion-free and $K$ is compact.

Our theorem has found applications in harmonic analysis. See [4].

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REFERENCES


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