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FUNCTIONS WITH A CONCAVE MODULUS OF CONTINUITY

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ABSTRACT. In [1], C. Goffman proved that, if σ is a modulus of continuity, then the set of all functions f in $C[0, 1]$ such that $m(\{x:f(x)=g(x)\})=0$ (m denotes Lebesgue measure) for all g in $C(\sigma)$, the set of all functions in $C[0, 1]$ having σ as a modulus of continuity, is residual in $C[0, 1]$. In the present article, we prove that, if σ is a concave modulus of continuity and $0<K<24^{-1}$, then the set of all functions f in $C(\sigma)$ such that $m(\{x:f(x)=g(x)\})=0$ for all g in $C(K\sigma)$ is residual in $C(\sigma)$. Using this result, we show that, if $0<\alpha<1$, then there are functions in $C[0, 1]$ which satisfy a Hölder condition of exponent α such that $m(\{x:f(x)=g(x)\})=0$ for all g in $C[0, 1]$ which satisfy a Hölder condition of exponent $>\alpha$.

1. By a modulus of continuity we mean a real valued function σ defined on $[0, 1]$ such that: (1) if $0<x\leqq y\leqq 1$, then $0<\sigma(x)\leqq\sigma(y)$, (2) σ is continuous at 0, and (3) $\sigma(0)=0$. We say that a function f in $C[0, 1]$ has modulus of continuity σ if

$$|f(y)-f(x)|\leqq\sigma(|y-x|) \tag{1.1}$$

for all x, y in $[0, 1]$. Here $C[0, 1]$ denotes the set of all continuous, real valued functions defined on $[0, 1]$. We denote by $C(\sigma)$ the set of all functions in $C[0, 1]$ having modulus of continuity σ.

In [1], C. Goffman proved the following statement.

1.1 THEOREM. *If σ is a modulus of continuity, then the set of all functions f in $C[0, 1]$ such that $\{x:f(x)=g(x)\}$ has Lebesgue measure 0 for all g in $C(\sigma)$ is residual in $C[0, 1]$ (i.e. its complement in $C[0, 1]$ is of the first category in $C[0, 1]$).*

This result leads naturally to the following question. If σ and μ are moduli of continuity, under what conditions is $U(\sigma, \mu)$, the set of all functions f in $C(\sigma)$ such that $m(\{x:f(x)=g(x)\})=0$ for all g in $C(\mu)$, residual in $C(\sigma)$? Here m denotes Lebesgue measure and $C(\sigma)$ is equipped with the restriction of the usual metric on $C[0, 1]$.

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In this paper we prove the following statement.

1.2 THEOREM. *If σ is a concave modulus of continuity and*

$$\limsup_{x\to 0}[\mu(x)\sigma^{-1}(x)] < 24^{-1},$$

then $U(\sigma, \mu)$ is residual in $C(\sigma)$.

The following corollary to 1.2 answers in the affirmative a question which Dr. Goffman suggested.

1.3 COROLLARY. *If $0<\alpha<1$, then there is a function f in $C[0, 1]$ such* that: (1) *f satisfies a Hölder condition of exponent α, and* (2) *if g in $C[0, 1]$ satisfies a Hölder condition of exponent $\beta>\alpha$, then $m(\{x: f(x)=g(x)\})=0$.*

2. If σ is a modulus of continuity and $A\subset[0, 1]$, we denote by $C(A, \sigma)$ the set of all functions $f: A\to R$ for which (1.1) holds for all x, y in A. We denote by $\mathrm{Lip}(A, \sigma)$ the set $\bigcup\{C(A, M\sigma): M>0\}$. If $f\in\mathrm{Lip}(A, \sigma)$, we let

$$\|f\|_\sigma = \inf\{M > 0: f\in C(A, M\sigma)\}.$$

Then

$$|f(y) - f(x)| \leqq \|f\|_\sigma\, \sigma(|y - x|) \tag{2.1}$$

for all x, y in A. We will denote $\mathrm{Lip}([0, 1], \sigma)$ by $\mathrm{Lip}(\sigma)$. As usual, for f in $C[0, 1]$, $\|f\|$ denotes $\max\{|f(x)|: x\in[0, 1]\}$.

A modulus of continuity σ is called subadditive if $\sigma(x+y)\leqq\sigma(x)+\sigma(y)$ for all x, y in $[0, 1]$ such that $x+y\in[0, 1]$. A modulus of continuity σ is called concave if, for any x, y in $[0, 1]$ such that $x<y$, we have $\sigma(t)\geqq T(t)$ for all t in $[x, y]$, where T is the linear function such that $T(x)=\sigma(x)$ and $T(y)=\sigma(y)$.

In the rest of this paper, σ will always denote a concave modulus of continuity and J will always denote a closed subinterval of $[0, 1]$. We denote the set of all nonnegative integers by N. For i, j in N, let $N(i,j)=\{k\in N: i\leqq k\leqq j\}$ and $N(j)=N(0,j)$. Let $N^+=N-\{0\}$.

Since quasi-linear functions play an important role in the proof of Theorem 1.2, it is convenient to have the following terminology. By a quasi-linear function on J we mean a function φ, defined and continuous on J, such that there is a partition $P=\{x_j: j\in N(n)\}$ of J for which $\varphi|[x_j, x_{j+1}]$ is linear for all j in $N(n-1)$. In this case, P is called a φ-partition of J. If $P=\{x_j: j\in N(n)\}$ is a partition of J and $(c_j)_{j\in N(n)}$ is a family of real numbers, then the quasi-linear function φ defined on J such that P is a φ-partition of J and $\varphi(x_i)=c_i$ for $i\in N(n)$ is called the quasi-linear function determined by $\{(x_j, c_j): j\in N(n)\}$. For any partition $P=\{x_j: j\in N(n)\}$ of J, we denote by $\|P\|$ the number $\max\{x_{j+1}-x_j: j\in N(n-1)\}$. Finally, the set of all quasi-linear functions defined on J is denoted by $Q(J)$.

3. We shall prove 1.2 via a sequence of lemmata.

3.1 LEMMA. *Suppose that* $P=\{x_j:j\in N(n)\}$ *is a partition of* J *and that* $(c_j)_{j\in N(n)}$ *is a family of real numbers such that*

$$|c_j - c_i| \leqq \sigma(|x_j - x_i|) \tag{3.1}$$

for all i, j *in* $N(n)$. *Let* φ *denote the quasi-linear function determined by* $\{(x_j, c_j):j\in N(n)\}$. *Then* $\varphi\in C(J, \sigma)$.

PROOF. Suppose $x, y\in J$. We may assume $\{x, y\}\cap P\neq\varnothing$. [If $\{x, y\}\cap P=\varnothing$, then there are x', y' in J such that $|y'-x'|=|y-x|$, $|\varphi(y')-\varphi(x')|\geqq|\varphi(y)-\varphi(x)|$, and $\{x', y'\}\cap P\neq\varnothing$.] Suppose $x<y$, $x=x_i$, and k in $N(n)$ is such that $x_k\leqq y\leqq x_{k+1}$. Then, since (3.1) holds, $\varphi(x_k)-\varphi(x_i)\leqq\sigma(x_k-x_i)$ and $\varphi(x_{k+1})-\varphi(x_i)\leqq\sigma(x_{k+1}-x_i)$. Since φ is linear on $[x_k, x_{k+1}]$ and σ is concave, $\varphi(y)-\varphi(x_i)\leqq\sigma(y-x_i)$. Similarly, $\varphi(y)-\varphi(x_i)\geqq-\sigma(y-x_i)$.

3.2 COROLLARY. *Suppose* $\varphi\in Q(J)$ *and* $P=\{x_j:j\in N(n)\}$ *is a* φ-*partition of* J. *Then*

$$\|\varphi\|_\sigma = \max\{|\varphi(x_j) - \varphi(x_i)|\,[\sigma(|x_j - x_i|)]^{-1}: i, j\in N(n), i\neq j\}. \tag{3.2}$$

PROOF. Let M denote the number on the right hand side of (3.2). Applying 3.1, with $c_j=\varphi(x_j)$ for $j\in N(n)$, we have $\|\varphi\|_\sigma\leqq M$. By (2.1), $\|\varphi\|_\sigma\geqq M$.

3.3 COROLLARY. $Q(J)\cap C(J, \sigma)$ *is dense in* $C(J, \sigma)$.

3.4 LEMMA. *Suppose* φ *is a linear function defined on* $J=[a, b]$. *If* $x, y\in J$ *and* $|y-x|\geqq\frac{1}{4}(b-a)$, *then*

$$|\varphi(y) - \varphi(x)| \geqq \tfrac{1}{4}\,\|\varphi\|_\sigma\,\sigma(|y - x|). \tag{3.3}$$

PROOF. If x, y are as hypothesized, then, using (3.2),

$$\begin{aligned}|\varphi(y) - \varphi(x)| &= \frac{|\varphi(b) - \varphi(a)|}{\sigma(b - a)}\,\frac{\sigma(b - a)}{\sigma(|y - x|)}\,\frac{|y - x|}{b - a}\,\sigma(|y - x|)\\ &\geqq \tfrac{1}{4}\,\|\varphi\|_\sigma\,\sigma(|y - x|).\end{aligned}$$

Notation. For x in $[0, 1]$ and $\delta>0$, let $I(x, \delta)=[x+\frac{1}{4}\delta, x+\frac{1}{2}\delta]$ and $J(x, \delta)=[x-\frac{1}{2}\delta, x-\frac{1}{4}\delta]$.

3.5 LEMMA. *Suppose* $h>0$, $0<L<(24)^{-1}$, *and* $J=[a, b]$. *There is* θ *in* $Q(J)$ *such that*: (1) $\theta(a)=\theta(b)=0$, (2) $0\leqq\theta(x)\leqq h$ *for all* x *in* J, (3) $\|\theta\|_\sigma\leqq 1-4L$, *and* (4) *there is* $\delta=\delta(\theta)>0$ *such that, if* $x\in J$, *then*

$$|\theta(y) - \theta(x)| \geqq 5L\sigma(|y - x|) \tag{3.4}$$

holds either for all y *in* $I(x, \delta)$ *or for all* y *in* $J(x, \delta)$.

PROOF. Choose positive numbers d, h' so that $\frac{1}{2}(b-a)d^{-1} \in N$, $h' \leqq h$, and

$$20L \leqq h'[\sigma(d)]^{-1} \leqq 1 - 4L.$$

Let $k = \frac{1}{2}(b-a)d^{-1}$, $x_j = a + jd$ for $j \in N(2k)$, $P = \{x_j : j \in N(2k)\}$, and θ be the quasi-linear function determined by

$$\{(x_{2j}, 0) : j \in N(k)\} \cup \{(x_{2j+1}, h') : j \in N(k-1)\}.$$

Then, by (3.2),

$$\begin{aligned} \|\theta\|_\sigma &= \max\{|\theta(x_j) - \theta(x_i)| \, [\sigma(|x_j - x_i|)]^{-1} : i, j \in N(2k), i \neq j\} \\ &= |\theta(x_1) - \theta(x_0)| \, [\sigma(x_1 - x_0)]^{-1} = h'[\sigma(d)]^{-1} \leqq 1 - 4L. \end{aligned}$$

And, by (3.3), if $x, y \in [x_j, x_{j+1}]$ and $|y-x| \geqq \frac{1}{4}d$, then

$$\begin{aligned} |\theta(y) - \theta(x)| &\geqq \tfrac{1}{4} \, \|\theta \mid [x_j, x_{j+1}]\|_\sigma \, \sigma(|y - x|) \\ &= \tfrac{1}{4}h'[\sigma(d)]^{-1}\sigma(|y - x|) \\ &\geqq 5L\sigma(|y - x|). \end{aligned}$$

Let $\delta = \frac{1}{4}d$. If $x \in [x_j, x_j + 2\delta]$, then $I(x, \delta) \subset [x_j, x_{j+1}]$. Hence (3.4) holds for all y in $I(x, \delta)$. Similarly, if $x \in [x_j + 2\delta, x_{j+1}]$, then (3.4) holds for all y in $J(x, \delta)$.

For n in N^+ and any positive numbers L, η such that $\eta < n^{-1}$, let $V(\sigma, L, n, \eta)$ denote the set of all f in $C(\sigma)$ for which there is a measurable subset $A = A(f)$ of $[0, 1]$ such that $m(A) > 1 - n^{-1}$ and, if $x \in A$, then there is $\delta = \delta(x) > 0$ such that $\eta < \delta < n^{-1}$ and

$$|f(y) - f(x)| \geqq L\sigma(|y - x|) \tag{3.5}$$

holds either for all y in $I(x, \delta)$ or for all y in $J(x, \delta)$. Let

$$V(\sigma, L, n) = \bigcup \{V(\sigma, L, n, \eta) : 0 < \eta < n^{-1}\}.$$

3.6 LEMMA. *Suppose* $0 < L < 24^{-1}$, $0 < \alpha < 1$, $\varepsilon > 0$, *and* $n \in N^+$. *If* $\varphi \in C(\alpha\sigma) \cap Q([0, 1])$, *then there is* ψ *in* $C(\sigma) \cap Q([0, 1])$ *and* $\eta > 0$ *such that* $\psi \in V(\sigma, L, n, \eta)$ *and* $\|\varphi - \psi\| \leqq \varepsilon$.

PROOF. Let $P = \{x_j : j \in N(q)\}$ be a φ-partition of $[0, 1]$ such that $\|P\| < n^{-1}$ and, for $j \in N(q-1)$,

$$|\varphi(x_{j+1}) - \varphi(x_j)| < \tfrac{1}{2}\varepsilon. \tag{3.6}$$

For j in $N(q-1)$, let $I_j = [x_j, x_{j+1}]$, $I'_j = [x_0, x_j]$, and $I''_j = [x_{j+1}, x_q]$.

Suppose, for some j in $N(q-1)$, we have defined ψ on I'_j so that: (j.1) $\psi \in Q(I'_j)$, (j.2) $\|\psi - \varphi | I'_j\| \leqq \varepsilon$, ($j$.3) $\psi(x_i) = \varphi(x_i)$ for all i in $N(j)$, (j.4) if λ_j is defined by letting

$$\begin{aligned} \lambda_j(x) &= \psi(x) \quad \text{if } x \in I'_j, \\ &= \varphi(x) \quad \text{if } x \in I_j \cup I''_j, \end{aligned}$$

then $\|\lambda_j\|_\sigma<1$, and (j.5) there are a measurable subset A_j of I'_j and $\eta_j>0$ such that $m(A_j)>m(I'_j)-j(qn)^{-1}$ and, if $x\in A_j$, then there is a δ such that $\eta_j<\delta<n^{-1}$ and (3.5), with f replaced by ψ, holds either for all y in $I(x,\delta)$ or for all y in $J(x,\delta)$.

Case 1. Suppose $|\varphi(x_{j+1})-\varphi(x_j)|\geqq 4L\sigma(x_{j+1}-x_j)$. Then $\|\varphi|I_j\|_\sigma\geqq 4L$. Let $\psi(x)=\varphi(x)$ for all x in I_j. Clearly $(j+1)$.1–$(j+1)$.4 are satisfied. Let $A_{j+1}=A_j\cup I_j$ and $\eta_{j+1}=\min(\eta_j,\frac{1}{4}(x_{j+1}-x_j))$. Then $(j+1)$.5 holds, since, if $x,y\in I_j$ and $|y-x|\geqq\frac{1}{4}(x_{j+1}-x_j)$, by 3.4,

$$|\varphi(y)-\varphi(x)|\geqq\tfrac{1}{4}\|\varphi\mid I_j\|_\sigma\,\sigma(|y-x|)\geqq L\sigma(|y-x|).$$

Case 2. Suppose $|\varphi(x_{j+1})-\varphi(x_j)|<4L\sigma(x_{j+1}-x_j)$. Assume that $\varphi(x_{j+1})\geqq\varphi(x_j)$. By ($j$.1), there is a ψ-partition $P_1=\{y_i:i\in N(p)\}$ of I'_j. Let $d=x_{j+1}-x_j$, $x_j^*=x_j+\frac{1}{2}d$, and

$$\begin{aligned} b_j=\min[&\{\psi(y_i)+\sigma(x_j^*-y_i):i\in N(p)\}\\ &\cup\{\varphi(x_i)+\sigma(x_i-x_j^*):i\in N(j+1,q)\}]. \end{aligned}$$

Then $b_j>\varphi(x_j^*)$ since $\|\lambda_j\|_\sigma<1$. Let a_j be a number such that

$$\varphi(x_j^*)<a_j<\min(b_j,\varphi(x_{j+1})+\tfrac{1}{2}\varepsilon) \tag{3.7}$$

and let γ denote the quasi-linear function determined by

$$\{(y_i,\psi(y_i)):i\in N(p)\}\cup\{(x_j^*,a_j)\}\cup\{(x_i,\varphi(x_i)):i\in N(j+1,q)\}.$$

Then $\|\gamma\|_\sigma<1$. Because of (3.1), it suffices to show that

$$|\gamma(y)-\gamma(x)|<\sigma(|y-x|) \tag{3.8}$$

whenever x,y are in $P_1\cup\{x_j^*\}\cup\{x_i:i\in N(j+1,q)\}$ and $x\neq y$. And (3.8) holds since (j.4) and (3.7) hold, and

$$\begin{aligned} \max[&\{\psi(y_i)-\sigma(x_j^*-y_i):i\in N(p)\}\\ &\cup\{\varphi(x_i)-\sigma(x_i-x_j^*):i\in N(j+1,q)\}]<\varphi(x_j^*). \end{aligned}$$

Now let d' be a number such that $0<d'<\frac{1}{2}\min((nq)^{-1},d)$ and let $h=\gamma(x_j+d)-\varphi(x_j+d)=2d'd^{-1}(a_j-\varphi(x_j^*))$. Then $h>0$ and, for all x in $[x_j+d,x_{j+1}-d]=I_j^*$,

$$\varphi(x)+h\leqq\gamma(x). \tag{3.9}$$

By 3.5, with $J=I_j^*$, there is θ_j in $Q(I_j^*)$ satisfying (1)–(4) of 3.5. Note that $\delta(\theta_j)<n^{-1}$. Define ψ on I_j by letting

$$\begin{aligned} \psi(x)&=\varphi(x)+\theta_j(x) &&\text{if } x\in I_j^*,\\ &=\varphi(x) &&\text{if } x\in I_j-I_j^*. \end{aligned}$$

Clearly $(j+1)$.1 and $(j+1)$.3 hold. Next, since (3.6) and (3.7) hold,

$(j+1).2$ holds. Now we verify that $(j+1).4$ holds. It follows from (3.9) and (2) of 3.5 that, for all x in I_j,

$$\varphi(x) \leqq \psi(x) \leqq \gamma(x). \tag{3.10}$$

Next, $\|\psi|I_j^*\|_\sigma<1$. For suppose $x, y \in I_j^*$ and $x\neq y$. Then

$$\begin{aligned}|\psi(y) - \psi(x)| &\leqq |\varphi(y) - \varphi(x)| + |\theta_j(y) - \theta_j(x)| \\ &\leqq \|\varphi \mid I_j\|_\sigma \, \sigma(|y - x|) + \|\theta_j\|_\sigma \, \sigma(|y - x|) \\ &< 4L\sigma(|y - x|) + (1 - 4L)\sigma(|y - x|) = \sigma(|y - x|).\end{aligned}$$

And $\|\psi|I_j\|_\sigma<1$. For suppose P_2 is a θ_j-partition of I_j^* and let $P_3=P_2\cup \{x_j, x_{j+1}\}$. Since $\|\psi|I_j^*\|_\sigma<1$, it suffices to show that (3.8), with γ replaced by ψ, holds whenever $x, y \in P_3$, $x\neq y$, and either $x \notin P_2$ or $y \notin P_2$. But, in this case, since (3.10) holds, $|\psi(y)-\psi(x)|\leqq|\gamma(y)-\gamma(x)|<\sigma(|y-x|)$. Now we shall show that $\|\lambda_{j+1}\|_\sigma<1$. Since $\|\lambda_{j+1}|I_j\|_\sigma<1$, it suffices to show that (3.8), with γ replaced λ_{j+1}, holds whenever $x, y \in [0, 1]$, $x<y$, and either $x \notin I_j$ or $y \notin I_j$. Suppose $x \in I_j'$ and $y \in I_j$. Then, since (3.10) holds, $\gamma(x_j)\leqq \lambda_{j+1}(y)\leqq\gamma(y)$. So there is z in $(x_j, y]$ such that $\gamma(z)=\lambda_{j+1}(y)$. Therefore

$$|\lambda_{j+1}(y) - \lambda_{j+1}(x)| = |\gamma(z) - \gamma(x)| < \sigma(|z - x|) \leqq \sigma(|y - x|).$$

A similar argument applies when $x \in I_j$ and $y \in I_j''$. It remains to show that $(j+1).5$ holds. To do this, let $A_{j+1}=A_j\cup I_j^*$ and $\eta_{j+1}=\min(\eta_j, \delta(\theta_j))$. Suppose $x \in I_j^*$ and that (3.4), with θ replaced by θ_j, holds for all y in $I(x, \delta(\theta_j))(J(x, \delta(\theta_j)))$. Then, if y is as specified,

$$\begin{aligned}|\psi(y) - \psi(x)| &\geqq |\theta_{j+1}(y) - \theta_{j+1}(x)| - |\varphi(y) - \varphi(x)| \\ &\geqq 5L\sigma(|y - x|) - \|\varphi \mid I_j\|_\sigma \, \sigma(|y - x|) \\ &\geqq 5L\sigma(|y - x|) - 4L\sigma(|y - x|) = L\sigma(|y - x|).\end{aligned}$$

It is clear that after q steps we have the required function ψ.

3.7 THEOREM. *If $0<M<24^{-1}$, then $U(\sigma, M\sigma)$ is residual in $C(\sigma)$.*

PROOF. Suppose $M<K<24^{-1}$. Since $C(\sigma)$ is a complete metric space, it suffices to show that:

(1) for each n in N^+, the interior of $V(\sigma, K, n)$ is dense in $C(\sigma)$, and
(2) $\bigcap \{V(\sigma, K, n): n \in N^+\}\subset U(\sigma, M\sigma)$.

PROOF OF (1). Suppose $n \in N^+$. First, it follows from 3.3 that the set $D=\{\varphi \in Q([0, 1]): \|\varphi\|_\sigma<1\}$ is dense in $C(\sigma)$. Fix L in $(K, 24^{-1})$. It follows from 3.6 that $V(\sigma, L, n)\cap D$ is dense in D. We shall show that $V(\sigma, L, n)$ is contained in the interior of $V(\sigma, K, n)$. Suppose $f\in V(\sigma, L, n, \eta)$. Let $d=\frac{1}{2}(L-K)\sigma(\frac{1}{4}\eta)$. We shall show that

$$\{g \in C(\sigma): \|f - g\| \leqq d\} \subset V(\sigma, K, n, \eta). \tag{3.11}$$

There is a measurable subset $A(f)$ of $[0, 1]$ such that $m(A(f))>1-n^{-1}$ and, if $x \in A(f)$, then there is $\delta(x,f)$ such that $\eta<\delta(x,f)<n^{-1}$ and (3.5) holds either for all y in $I(x, \delta(x,f))$ or for all y in $J(x, \delta(x,f))$. Suppose $g \in C(\sigma)$ and $\|f-g\| \leqq d$. Let $A(g)=A(f)$ and, for x in $A(g)$, let $\delta(x, g)=\delta(x,f)$. If (3.5) holds for all y in $I(x, \delta(x,f))$ and $y \in I(x, \delta(x, g))$, then, since $\sigma(|y-x|) \geqq \sigma(\frac{1}{4}\eta)$,

$$\begin{aligned} |g(y) - g(x)| &\geqq |f(y) - f(x)| - 2d \\ &\geqq L\sigma(|y - x|) - (L - K)\sigma(|y - x|) = K\sigma(|y - x|). \end{aligned} \tag{3.12}$$

Similarily, if (3.5) holds for all y in $J(x, \delta(x,f))$, then (3.12) holds for all y in $J(x, \delta(x, g))$. Hence (3.11) holds.

Proof of (2). Suppose $f \in \bigcap \{V(\sigma, K, n): n \in N^+\}$. Then there is a sequence $(A_n)_{n \in N^+}$ of measurable subsets of $[0, 1]$ such that $\lim_{n\to\infty} m(A_n)=1$ and, if $x \in A_n$, then there is $\delta(x, n)>0$ such that $\delta(x, n)<n^{-1}$ and (3.5), with L replaced by K, holds either for all y in $I(x, \delta(x, n))$ or for all y in $J(x, \delta(x, n))$. Let

$$A = \bigcap \{\bigcup \{A_k : k \in N^+, k \geqq n\} : n \in N^+\}.$$

Then $m(A)=1$.

Suppose, now, that $g \in C(M\sigma)$ and let $X=\{x: f(x)=g(x)\}$. Suppose $x \in A \cap X$. We shall show that the lower metric density of X at x is <1. There is an increasing sequence $(k(n))_{n \in N^+}$ of positive integers such that $x \in \bigcap \{A_{k(n)} : n \in N^+\}$. For n in N^+, let

$$I_n = [x - \tfrac{1}{2}\delta(x, k(n)), x + \tfrac{1}{2}\delta(x, k(n))].$$

If $n \in N^+$, then

$$m(\{y: |f(y) - f(x)| \geqq K\sigma(|y - x|)\} \cap I_n) \geqq \tfrac{1}{4}m(I_n).$$

Since

$$\{y: |f(y) - f(x)| \geqq K\sigma(|y - x|)\} \subset \{y: f(y) \neq g(y)\},$$

$4m(X \cap I_n) \leqq 3m(I_n)$. And, $\lim_{n\to\infty} m(I_n)=\lim_{n\to\infty} \delta(x, k(n))=0$. Thus the lower metric density of X at x is $\leqq \frac{3}{4}$. Hence, by the Lebesgue density theorem, $m(A \cap X)=0$ and $f \in U(\sigma, M\sigma)$.

Proof of Theorem 1.2. Choose M so that

$$\limsup_{x\to 0} [\mu(x)\sigma^{-1}(x)] < M < 24^{-1}.$$

We shall show that $U(\sigma, M\sigma) \subset U(\sigma, \mu)$. First, there is $\delta>0$ such that $\mu(x) \leqq M\sigma(x)$ whenever $0 \leqq x \leqq \delta$. Suppose $f \in U(\sigma, M\sigma)$, $g \in C(\mu)$, and J is a closed subinterval of $[0, 1]$ of length $\leqq \delta$. Then $g|J \in C(J, M\sigma)$. Let h in $C(M\sigma)$ be such that $h|J=g|J$. Then $m(\{x: f(x)=h(x)\})=0$ since $f \in U(\sigma, M\sigma)$; hence $m(\{x: f(x)=g(x)\} \cap J)=0$. Therefore $f \in U(\sigma, \mu)$.

3.8 THEOREM. *Let $V(\sigma)$ denote the set of all f in $C(\sigma)$ such that $m(\{x: f(x)=g(x)\})=0$ for all g in $\bigcup \{\mathrm{Lip}(\mu): \lim_{x\to 0} [\mu(x)\sigma^{-1}(x)]=0\}$. Then $V(\sigma)$ is residual in $C(\sigma)$.*

PROOF. If $0<M<24^{-1}$ and $\lim_{x\to 0} [\mu(x)\sigma^{-1}(x)]=0$, then $U(\sigma, M\sigma)\subset \bigcap \{U(\sigma, K\mu): K>0\}$. Hence $U(\sigma, M\sigma)\subset V(\sigma)$.

3.9 DEFINITION. For α in $(0, 1]$, let $\sigma_\alpha(t)=t^\alpha$ for all t in $[0, 1]$. For any subset A of $[0, 1]$, let $\mathrm{Lip}_\alpha(A)=\mathrm{Lip}(A, \sigma_\alpha)$. Denote $\mathrm{Lip}_\alpha([0, 1])$ by Lip_α.

In the rest of this paper α will always denote a number in $(0, 1)$.

3.10 COROLLARY. *Suppose $M>0$. Let $W(\alpha, M)$ be the set of all f in $C(M\sigma_\alpha)$ such that $m(\{x: f(x)=g(x)\})=0$ for all g in $\bigcup \{\mathrm{Lip}_\beta: \alpha<\beta\leqq 1\}$. Then $W(\alpha, M)$ is residual in $C(M\sigma_\alpha)$.*

PROOF. This follows from 3.8 since $M\sigma_\alpha$ is a concave modulus of continuity.

3.11 THEOREM. *Suppose μ is a subadditive modulus of continuity, $A\subset [0, 1]$, and $f\in C(A, \mu)$. Then there is f^* in $C(\mu)$ such that $f^*|A=f$.*

The proof of 3.11 is the same as in the special case where $\mu=M\sigma_\beta$ for some $\beta\in (0, 1]$ and $M>0$.

3.12 COROLLARY. *Suppose σ, μ satisfy the hypothesis of 1.2. If $f\in U(\sigma, \mu)$ and A is a subset of $[0, 1]$ such that $f|A\in C(A, \mu)$, then $m(A)=0$.*

PROOF. This follows from 1.2 and 3.11, since any concave modulus of continuity is subadditive.

In [2], W. S. Loud constructed a function f in Lip_α such that, if $\alpha<\beta\leqq 1$ and J is a closed subinterval of $[0, 1]$, then $f|J\notin \mathrm{Lip}_\beta(J)$. The following statement shows that, in a certain sense, most functions in Lip_α have a somewhat stronger property.

3.13 COROLLARY. *Suppose $M>0$. Let $W(\alpha, M)$ denote the set defined in 3.10. If $f\in W(\alpha, M)$ and A is a subset of $[0, 1]$ such that $f|A\in \bigcup \{\mathrm{Lip}_\beta(A): \alpha<\beta\leqq 1\}$, then $m(A)=0$.*

Denote the set of all real numbers by R. In [2], W. S. Loud constructed a function $f: R\to R$ with the following property: There are positive numbers M, K such that $K<M$ for which $|f(y)-f(x)|\leqq M|y-x|^\alpha$ for all x, y in R and

$$\limsup_{h\to 0} |[f(x+h)-f(x)]h^{-\alpha}|\geqq K$$

for all x in R.

3.14 DEFINITION. Suppose F is a measurable function defined on $(-a, 0)\cup(0, a)$ for some $a>0$. The approximate upper limit of F at 0, denoted by lim sup $\mathrm{ap}_{x\to 0} F(x)$ is defined to be the infimum of the set of real numbers y such that $\{x : F(x) \leqq y\}$ has metric density 1 at 0.

Note that $\limsup \mathrm{ap}_{x\to 0} F(x) \leqq \limsup_{x\to 0} F(x)$.

3.15 COROLLARY. *Suppose* $0 < K < 24^{-1}M$. *Let* $S(\alpha, M, K)$ *denote the set of all functions* f *in* $C(M\sigma_\alpha)$ *such that*

$$\limsup_{h\to 0} \mathrm{ap}\, |[f(x+h) - f(x)]h^{-\alpha}| \geqq K$$

for almost all x *in* $[0, 1]$. *Then* $S(\alpha, M, K)$ *is residual in* $C(M\sigma_\alpha)$.

PROOF. This follows from the proof of 3.7, since

$$\bigcap \{V(M\sigma_\alpha, KM^{-1}, n) : n \in N^+\} \subset S(\alpha, M, K).$$

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