A SIMPLE ALTERNATIVE PROBLEM FOR FINDING PERIODIC SOLUTIONS OF SECOND ORDER ORDINARY DIFFERENTIAL SYSTEMS

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Abstract. Existence of solutions for $x'' = f(t, x, x')$, $x(0) = x(1)$, $x'(0) = x'(1)$ are proven by considering a simple alternative problem to which Leray-Schauder degree arguments can be directly applied.

1. Introduction. In this paper, we consider the existence of solutions to the periodic boundary value problem (PBVP)

\begin{align}
(1) & \quad x'' = f(t, x, x'), \\
(2) & \quad x(0) = x(1), \quad x'(0) = x'(1).
\end{align}

Knobloch [4], Mawhin [5], Schmitt [6], and Bebernes and Schmitt [1] have recently considered this problem using degree-theoretic arguments—either finite or infinite dimensional.

Using only the basic properties of Leray-Schauder degree and applying these degree arguments to a simple alternative problem associated with (1)-(2), we obtain in this paper a single basic result (Theorem 2.1) which contains and in some cases permits slight generalizations of most of the results of the above mentioned papers.

2. The basic theorem. Let $I = [0, 1]$, $\mathbb{R}^n$ be $n$-dimensional Euclidean space with Euclidean norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$, and let $D \subset I \times \mathbb{R}^n \times \mathbb{R}^n$ be a bounded open set in the relative topology of $I \times \mathbb{R}^n \times \mathbb{R}^n$ containing \{(t, 0, 0) : t \in I\}. Let $F : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function and consider

\begin{equation}
(3) \quad x'' = F(t, x, x').
\end{equation}

For each $\lambda \in [0, 1]$, associate with (3) the equation

\begin{equation}
(4) \quad x'' = \lambda F(t, x, x') + (1 - \lambda)x.
\end{equation}

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and assume:

(H) If \( x(t) \) is a solution of (4)-(2), then \( (t, x(t), x'(t)) \in D \) for all \( t \in I \) or there exists \( \tau \in I \) such that \( (\tau, x(\tau), x'(\tau)) \notin \bar{D} \).

**Theorem 2.1.** The periodic boundary value problem (3)-(2) has at least one solution such that \( (t, x(t), x'(t)) \in D \) for all \( t \in I \).

**Proof.** The periodic boundary value problem

\[
(5) \quad x'' - x = 0, \quad x(0) = x(1), \quad x'(0) = x'(1)
\]

has no nontrivial solutions. Let \( H(t, x, x') = F(t, x, x') - x \), then \( x(t) \) is a solution of (4)-(2) if and only if \( x(t) \) is a solution of

\[
(6) \quad x(t) = \lambda \int_0^1 G(t, s)H(s, x(s), x'(s)) \, ds
\]

where \( G(t, s) \) is the unique Green's function for (5).

Let \( B = \{ x \in C^1[0, 1] : x(0) = x(1), x'(0) = x'(1) \} \) with norm

\[
|x| = \max_I \| x(t) \| + \max_I \| x'(t) \|
\]

be the Banach space under consideration, and define

\( \Omega = \{ y \in B : (t, y(t), y'(t)) \in D \text{ for all } t \in I \} \).

Note that \( \Omega \) is a bounded open subset of \( B \).

Define the map \( T : \bar{\Omega} \rightarrow B \) where \( \bar{\Omega} \) is the closure of \( \Omega \) by

\[
(7) \quad (Ty)(t) = \int_0^1 G(t, s)H(s, y(s), y'(s)) \, ds.
\]

By standard arguments, \( T(\bar{\Omega}) \subseteq B \), \( T \) is continuous, and \( \text{cl}(T(\bar{\Omega})) \) is compact in \( B \).

If \( 0 \notin (I - \lambda T)(\partial \Omega) \) where \( \partial \Omega \) is the boundary of \( \Omega \) for all \( \lambda \in [0, 1] \), then by the invariance under compact homotopy property of the Leray-Schauder degree [7, p. 92], the degree \( \text{deg}(I - \lambda T, \Omega, 0) = \text{constant} \) for all \( \lambda \in [0, 1] \). That \( 0 \in (I - \lambda T)(\partial \Omega) \) is equivalent to the existence of a solution \( x(t) \) of the PBVP (4)-(2) with \( (t, x(t), x'(t)) \in \bar{D} \) for all \( t \in I \) and \( (t, x(t), x'(t)) \in \partial D \) for some \( t \in I \); but by assumption (H) there exists no such solution \( x(t) \) of (4)-(2) with \( (t, x(t), x'(t)) \in \bar{D} \) for all \( t \in I \) and \( (t, x(t), x'(t)) \in \partial \bar{D} \) for some \( t \in I \). Hence, \( \text{deg}(I - T, \Omega, 0) = \text{deg}(I, \Omega, 0) = 1 \).

By the existence property of the Leray-Schauder degree [7, p. 88], there exists \( x \in \Omega \) such that \( (I - T)x = 0 \). This means that there exists a solution \( x(t) \) of the PBVP (3)-(2) with \( (t, x(t), x'(t)) \in D \) for all \( t \in I \).

### 3. Applications of the basic theorem.

In this section, we illustrate how Theorem 2.1 can be used to prove existence results for PBVP (1)-(2).
The first result is known (e.g., [1], [4], or [5]), but it well illustrates the power of our basic theorem.

**Theorem 3.1.** If \( f(t, x, x') \) is continuous on \( E_R = \{(t, x, x'): t \in I, \|x\| < R, \|x'\| < \infty\} \) and satisfies:

(8) \( \|x'\|^2 + \langle x, f(t, x, x') \rangle > 0 \) for all \( (t, x, x') \in E_R \) provided \( \|x\| = R \) and \( \langle x, x' \rangle = 0 \);

(9) \( \|f(t, x, x')\| \leq \varphi(\|x'\|) \) for all \( (t, x, x') \in E_R \) where \( \varphi \) is a positive continuous function on \([0, \infty)\) with \( \int_0^\infty s \varphi(s) \, ds = +\infty \);

(10) there exists \( \kappa \geq 0, K \geq 0 \) such that

\[
\|f(t, x, x')\| \leq 2\kappa(\|x'\|^2 + \langle x, f(t, x, x') \rangle) + K \quad \text{for all} \quad (t, x, x') \in E_R;
\]

then there exists a solution \( x(t) \) of the PBVP (1)-(2) with \( (t, x(t), x'(t)) \in E_R \).

**Proof.** Let \( \delta_M(s) \) be a continuous function on \([0, \infty)\) with \( \delta_M(s) = 1 \) on \([0, M]\) and \( \delta_M(s) = 0 \) for \( s \geq 2M \) where \( M = M(\kappa, K, R) \) is the Nagumo-Hartman bound (see Hartman [3, p. 429]).

Define

\[
F(t, x, x') = \delta_M(\|x'\|)f(t, x, x') \quad \text{on} \quad E_R, \quad \text{and} \quad F(t, x, x') = (R/\|x\|)F(t, Rx/\|x\|, x') \quad \text{if} \quad \|x\| \geq R.
\]

Then \( F(t, x, x') \) is continuous and bounded on \( I \times \mathbb{R}_+ \times \mathbb{R}_+ \) and satisfies (8) provided \( \|x\| \geq R \) and \( \langle x, x' \rangle = 0 \), (9), and (10) for all \( (t, x, x') \in I \times \mathbb{R}_+ \times \mathbb{R}_+ \).

The proof will be completed by showing that there can be constructed an open bounded set \( D \subseteq I \times \mathbb{R}_+ \times \mathbb{R}_+ \) containing \( \{(t, 0, 0): t \in I\} \) such that solutions of PBVP (4)-(2) satisfy hypothesis (H) relative to \( D \).

For each \( \lambda \in [0, 1] \), let \( F_\lambda(t, x, x') = \lambda F(t, x, x') + (1-\lambda)F(t, x, x') \) where \( F \) is defined as above. Then for all \( \lambda \in [0, 1] \) and all \( (t, x, x') \in I \times \mathbb{R}_+ \times \mathbb{R}_+ \),

\[
\|x'\|^2 + \langle x, F_\lambda(t, x, x') \rangle > 0 \quad \text{provided} \quad \|x\| \geq R
\]

and \( \langle x, x' \rangle = 0 \). Let \( x(t) \) be any solution of (4)-(2). Define \( u(t) = \|x(t)\|^2 = \langle x(t), x(t) \rangle \). Because \( u(t) \) satisfies the periodic boundary conditions (2), \( u(t) \) can assume its maximum at \( t_0 \in I \) only if \( u(t_0) = 0 \). Claim \( u(t) < R^2 \) for all \( t \in I \). Assume not; then there exists \( t_0 \in I \) at which \( u(t) \) assumes its maximum with \( u(t_0) \geq R^2 \), \( u'(t_0) = 0 \), and \( u''(t_0) \leq 0 \). But (11) implies that \( u''(t_0) > 0 \) which is a contradiction. Hence, \( \|x(t)\| < R \) for all \( t \in I \). For \( (t, x, x') \in E_R \) and \( \lambda \in [0, 1] \), \( F_\lambda(t, x, x') \) is bounded which implies that \( F_\lambda(t, x, x') \) satisfies a Nagumo-Hartman condition (conditions (9) and (10) with \( \varphi = 0 \) and a \( K' \) in general different from \( K \) and \( \varphi(s) = K' \)). Hence, there exists an \( M' > 0 \) such that if \( x(t) \) is any solution of (4) on \( I \) with \( \|x(t)\| < R \), then \( \|x'(t)\| < M' \).
Define \( D = \{(t, x, x') : t \in I, \|x\| < R, \|x'\| < M'\} \). From the observations made above it is immediate that solutions of (4)-(2) satisfy (H) relative to \( D \). By Theorem 2.1, the PBVP (3)-(2) has a solution \( x(t) \) with \( \|x(t)\| < R \). Since \( F(t, x, x') \) satisfies (9) and (10), \( \|x'(t)\| < M \) on \([0, 1]\) which implies that \( x(t) \) is a solution of PBVP (1)-(2) on \( I \) with \((t, x(t), x'(t)) \in E_2\).

Equality can be permitted in (8) by an approximating argument like the one given in [3, p. 433].

The preceding theorem can be generalized by replacing \( \|x\|^2 \) by a function \( V(t, x) \) which plays essentially the same role. In so doing, we obtain results similar to those obtained by Knobloch [4] and Mawhin [5].

Assume \( f(t, x, x') : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) is continuous and let \( R^+ \) denote the nonnegative reals.

**Definition.** Let \( V \in C^2(I \times \mathbb{R}^n \times \mathbb{R}^n, R^+) \) be such that:

(a) there exists \( R > 0 \) such that \( \Phi = \{x \in \mathbb{R}^n : V(t, x) < R, t \in I\} \) is bounded,

(b) \( U(t, x, x') = V_{xx}(t, x)x' + 2(V_{xt}(t, x), x') + \langle V(t, x), x' \rangle \geq 0 \),

(c) \( V_x(t, x) = U(t, x, x') + \langle V(t, x, x'), f(t, x, x') \rangle > 0 \) provided \( V(t, x) = R \) and \( V_x(t, x) = \langle V(t, x), x' \rangle = 0 \),

(d) \( \langle V_x(t, x), x' \rangle > 0 \) for all \((t, x)\) such that \( V(t, x) = R \),

(e) \( V(0, x) = V(1, x), V_i(0, x) + \langle V(t, x), x' \rangle \geq V_i(1, x) + \langle V(t, x), x' \rangle \).

Any such \( V \) is called a bounding Lyapunov function relative to (1).

**Theorem 3.2.** If \( V \) is a bounding Lyapunov function for (1), then for every \( \lambda \in [0, 1] \) every solution \( x(t) \) of the PBVP:

\[
V(t, x(t)) = R \quad \text{for some} \quad t \in I \\
V(t, x(t)) < R \quad \text{for all} \quad t \in I
\]

is such that \( V(\tau, x(\tau)) > R \) for some \( \tau \in I \) or \( V(t, x(t)) < R \) for all \( t \in I \).

**Proof.** Let \( x(t) \) be any solution of the PBVP (12)-(2) and let \( m(t) = V(t, x(t)) \), then \( m'(t) = V_i(t, x(t)) + \langle V_x(t, x(t)), x'(t) \rangle \) and

\[
m''(t) = U(t, x(t), x'(t)) + \langle V_x(t, x(t)), f_x(t, x(t), x'(t)) \rangle.
\]

By (b), (c), and (d), \( m''(t) > 0 \) if \( V(t, x(t)) = R \) and \( V_i(t, x(t)) + \langle V_x(t, x(t)), x'(t) \rangle = 0 \). If there exists \( \tau \in I \) such that \( m(\tau) > R \), we are through.

Assume \( m(t) \leq R \) for all \( t \in [0, 1] \). If there exists \( t_0 \in I \) such that \( m(t_0) = R \), then \( m'(t_0) = 0 \) and \( m''(t_0) \leq 0 \) since \( m(0) = m(1) \) and \( m'(0) \geq m'(1) \) by (e).

But this is impossible by the observation made above that \( m''(t_0) > 0 \). Hence, \( m(t) < R \) on \( I \) and the conclusion of the theorem follows.

Our next theorem is similar to Theorem 6.1 [5].
Theorem 3.3. If $V$ is a positive definite bounding Lyapunov function relative to (1) and if there exists $S>0$ such that for any $\lambda \in [0, 1]$ any solution $x(t)$ of PBVP (12)-(2) with $V(t, x(t)) < R$ on $I$ satisfies $\|x'(t)\| < S$ for $t \in I$, then PBVP (1)-(2) has at least one solution $x(t)$ with $V(t, x(t)) < R$.

Proof. Let $D=\{(t, x, x') : t \in I, \ V(t, x) < R, \ \|x'\| < S\}$. By Theorem 3.2, solutions of (12)-(2) satisfy (H) relative to $D$. Hence, by Theorem 1.2, the conclusion follows.

There are several ways of ensuring the a priori bound condition on the derivative of solutions of (12)-(2) and hence we have the following corollaries.

Corollary 3.4. If $V$ is a bounding positive definite Lyapunov function for (1) and if $f(t, x, x')$ satisfies (9) and (10) for all $t \in I, x \in \Phi, \ \|x'\| < \infty$, then PBVP (1)-(2) has a solution $x(t) \in \Phi$ for all $t \in I$.

Corollary 3.5. If $V$ is a bounding positive definite Lyapunov function for (1), if $f(t, x, x')$ satisfies (9) for all $t \in I, x \in \Phi, \ \|x'\| < \infty$, and if there exists $\beta \geq 0, L \geq 0$ such that

\[ \|f(t, x, x')\| \leq \beta(U(t, x, x') + \langle V_a(t, x), f(t, x, x') \rangle) + L \]

for all $t \in I, x \in \Phi$, and $\|x'\| \leq \infty$, then PBVP (1)-(2) has a solution $x(t) \in \Phi$ for all $t \in I$.

Corollary 3.6. If $V$ is a bounding positive definite Lyapunov function for (1), if $f(t, x, x')$ satisfies (9) for all $t \in I, x \in \Phi, \ \|x'\| < \infty$, and if there exists a function $\rho(t) \in C^2(I)$ such that

\[ \|f(t, x, x')\| \leq \rho'(t) \quad \text{for all } t \in I, x \in \Phi, \ \|x'\| < \infty, \]

then PBVP (1)-(2) has a solution $x(t) \in \Phi$ for all $t \in I$.

4. Further consequences. In this section, we present two further applications of Theorem 2.1. The first theorem presented shows that the bounding set $\Phi$ need not be given in terms of a bounding Lyapunov function. Assume $f(t, x, x') : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

Theorem 4.1. Let $G$ be a bounded convex open set in $\mathbb{R}^n$ containing 0 and assume there is a function $N : \partial G \to \mathbb{R}^n$ satisfying:

\[ \langle N(x), x \rangle > 0 \quad \text{for all } x \in \partial G, \]

\[ G \subseteq \{ y : \langle N(x), y - x \rangle \leq 0 \quad \text{for each } x \in \partial G \}, \]

\[ \langle N(x), f(t, x, x') \rangle > 0 \quad \text{for all } t \in I, x \in \partial G, \]

\[ x' \text{ with } \langle N(x), x' \rangle = 0, \]

then for every $\lambda \in [0, 1]$ every solution $x(t)$ of (12)-(2) is such that $x(\tau) \notin G$ for some $\tau \in I$ or $x(t) \in G$ for all $t \in I$. 

Remark. Conditions (16) and (17) say that $N(x)$ is an outer normal for $G$. Gustafson and Schmitt [2] have used a similar outer normal condition to study existence of periodic solutions for delay differential equations.

Proof. Let $x(t)$ be any solution of (12)-(2). If $x(t) \notin G$ for some $t \in I$, we are through so assume $x(t) \in G$ for all $t \in I$.

If $x(t_0) \in \partial G$ for some $t_0 \in I$, we may assume $t_0 \in [0, 1)$. By (16) and (18), $\langle N(x(t_0)), f'(t_0, x(t_0), x'(t_0)) \rangle > 0$ and hence there is an $h > 0$ such that $\langle N(x(t_0)), x'(t) \rangle > 0$ for all $t \in [t_0, t_0 + h)$. Since $x(t) \in G$, $\langle N(x(t_0)), x'(t_0) \rangle = 0$. Looking at the Taylor expansion for $x(t)$, we have immediately that

$$\langle N(x(t_0)), x(t) - x(t_0) \rangle = \langle N(x(t_0)), x'(t_0) \rangle + \frac{1}{2} (t - t_0)^2 \langle N(x(t_0)), y(\xi) \rangle$$

where $y(\xi) = (x'_1(\xi), \ldots, x'_n(\xi))$ and $t_0 < \xi_i < t < t_0 + h$ for all $i = 1, \ldots, n$. From this, $\langle N(x(t_0)), x(t) - x(t_0) \rangle > 0$ meaning that $x(t) \notin G$, which is a contradiction.

Our existence theorem then follows.

Theorem 4.2. If $G$ is a bounded convex open set in $\mathbb{R}^n$ containing 0, if there is a function $N: \partial G \to \mathbb{R}^n$ satisfying (16), (17), and (18), and if there exists $S > 0$ such that for any $x(t)$ of PBVP (12)-(2) with $x(t) \in G$ for all $t \in I$ satisfies $\|x'(t)\| < S$ for $t \in I$, then PBVP (1)-(2) has at least one solution with $x(t) \in G$ for all $t \in I$.

Proof. Let $D = \{(t, x, x') : t \in I, x \in G, \|x'\| < S\}$. By Theorem 4.1 solutions of (11)-(2) satisfy (H) relative to $D$. Result then follows immediately from Theorem 2.1.

Remark. One can state corollaries of the above theorem analogous to Corollaries 3.4, 3.5, and 3.6.

In $\mathbb{R}^n$, let $x \leq y$ if and only if $x_i \leq y_i$, $1 \leq i \leq n$, and $x < y$ if and only if $x_i < y_i$, $1 \leq i \leq n$.

Let $f(t, x, x')$ be continuous on $\{(t, x, x') : t \in I, x(t) \leq x(t') \leq \beta(t), x' \in \mathbb{R}^n\}$ where $\alpha, \beta: I \to \mathbb{R}^n$, $\alpha(t) < \beta(t)$ are twice continuously differentiable with

$$\begin{align*}
\alpha(0) = \alpha(1), & \quad \beta(0) = \beta(1), & \quad \alpha'(0) \geq \alpha'(1), & \quad \beta'(0) \leq \beta'(1).
\end{align*}$$

Assume also that $\alpha, \beta$ are strict lower, upper solutions of (1), i.e.,

$$\begin{align*}
\alpha''(t) & > f(t, x_1, \ldots, x_{i-1}, \alpha(t), x_{i+1}, \ldots, x_n, x'_1, \ldots, x'_n), \\
\beta''(t) & < f(t, x_1, \ldots, x_{i-1}, \beta(t), x_{i+1}, \ldots, x_n, x'_1, \ldots, x'_n).
\end{align*}$$

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We now can state our final result.

**Theorem 4.3.** If \( f \) is continuous on \( \{(t, x, x'): t \in I, \alpha(t) \leq x \leq \beta(t), x \in \mathbb{R}^n\} \) where \( \alpha, \beta \) are strict periodic lower, upper solutions of (1) satisfying (19), (20), and (21), and if there exists \( S > 0 \) such that for any \( \lambda \in [0, 1] \) any solution \( x(t) \) of (12)-(2) with \( \alpha(t) \leq x(t) \leq \beta(t) \) on \( I \) satisfies \( \|x'(t)\| < S \) then PBVP (1)-(2) has a solution \( x(t) \) with \( \alpha(t) < x(t) < \beta(t) \).

The proof is similar to those previously given and is for this reason omitted. By a proper modification of \( f(t, x, x') \), condition (21) can be dropped and equality can be permitted in (20). With that observation, we have a generalization of Theorem 4.1 in [1].

**References**


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