A SIMPLE ALTERNATIVE PROBLEM FOR FINDING PERIODIC SOLUTIONS OF SECOND ORDER ORDINARY DIFFERENTIAL SYSTEMS\textsuperscript{1}

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Abstract. Existence of solutions for $x'' = f(t, x, x')$, $x(0) = x(1)$, $x'(0) = x'(1)$ are proven by considering a simple alternative problem to which Leray-Schauder degree arguments can be directly applied.

1. Introduction. In this paper, we consider the existence of solutions to the periodic boundary value problem (PBVP)

(1) $x'' = f(t, x, x')$,

(2) $x(0) = x(1), \quad x'(0) = x'(1)$.

Knobloch [4], Mawhin [5], Schmitt [6], and Bebernes and Schmitt [1] have recently considered this problem using degree-theoretic arguments—either finite or infinite dimensional.

Using only the basic properties of Leray-Schauder degree and applying these degree arguments to a simple alternative problem associated with (1)-(2), we obtain in this paper a single basic result (Theorem 2.1) which contains and in some cases permits slight generalizations of most of the results of the above mentioned papers.

2. The basic theorem. Let $I = [0, 1]$, $\mathbb{R}^n$ be $n$-dimensional Euclidean space with Euclidean norm $\|\cdot\|$ and inner product $(\cdot, \cdot)$, and let $D \subseteq I \times \mathbb{R}^n \times \mathbb{R}^n$ be a bounded open set in the relative topology of $I \times \mathbb{R}^n \times \mathbb{R}^n$ containing $\{(t, 0, 0) : t \in I\}$. Let $F: I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function and consider

(3) $x'' = F(t, x, x').$

For each $\lambda \in [0, 1]$, associate with (3) the equation

(4) $x'' = \lambda F(t, x, x') + (1 - \lambda)x.$

Received by the editors April 6, 1973.


Key words and phrases. Periodic boundary value problems, alternative problems, Leray-Schauder degree, Nagumo-Hartman condition, Lyapunov-like functions.

\textsuperscript{1} This research was supported by the U.S. Air Force under Grant AFOSR-72-2379.

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and assume:

(H) If \( x(t) \) is a solution of (4)-(2), then \( (t, x(t), x'(t)) \in D \) for all \( t \in I \) or there exists \( \tau \in I \) such that \( (\tau, x(\tau), x'(\tau)) \notin \bar{D} \).

**Theorem 2.1.** The periodic boundary value problem (3)-(2) has at least one solution such that \( (t, x(t), x'(t)) \in D \) for all \( t \in I \).

**Proof.** The periodic boundary value problem

\[
(5) \quad x'' - x = 0, \quad x(0) = x(1), \quad x'(0) = x'(1)
\]

has no nontrivial solutions. Let \( H(t, x, x') = F(t, x, x') - x \), then \( x(t) \) is a solution of (4)-(2) if and only if

\[
(6) \quad x(t) = \lambda \int_0^1 G(t, s)H(s, x(s), x'(s)) \, ds
\]

where \( G(t, s) \) is the unique Green's function for (5).

Let \( B = \{ x \in C'[0,1] : x(0) = x(1), x'(0) = x'(1) \} \) with norm

\[
|x| = \max_i \| x(i) \| + \max_i \| x'(i) \|
\]

be the Banach space under consideration, and define

\[
\Omega = \{ y \in B : (t, y(t), y'(t)) \in D \text{ for all } t \in I \}.
\]

Note that \( \Omega \) is a bounded open subset of \( B \).

Define the map \( T : \Omega \to B \) where \( \Omega \) is the closure of \( \Omega \) by

\[
(7) \quad (Ty)(t) = \int_0^1 G(t, s)H(s, y(s), y'(s)) \, ds.
\]

By standard arguments, \( T(\bar{\Omega}) \subset B \), \( T \) is continuous, and \( \text{cl}(T(\bar{\Omega})) \) is compact in \( B \).

If \( 0 \notin (I-\lambda T)(\partial \Omega) \) where \( \partial \Omega \) is the boundary of \( \Omega \) for all \( \lambda \in [0,1] \), then by the invariance under compact homotopy property of the Leray-Schauder degree [7, p. 92], the degree \( \text{deg}(I-\lambda T, \Omega, 0) = \text{constant} \) for all \( \lambda \in [0,1] \). That \( 0 \notin (I-\lambda T)(\partial \Omega) \) is equivalent to the existence of a solution \( x(t) \) of the PBVP (4)-(2) with \( (t, x(t), x'(t)) \in \bar{D} \) for all \( t \in I \) and \( (t, x(t), x'(t)) \in \partial D \) for some \( t \in I \); but by assumption (H) there exists no such solution \( x(t) \) of (4)-(2) with \( (t, x(t), x'(t)) \in \bar{D} \) for all \( t \in I \) and \( (t, x(t), x'(t)) \in \partial D \) for some \( t \in I \). Hence, \( \text{deg}(I-T, \Omega, 0) = \text{deg}(I, \Omega, 0) = 1 \). By the existence property of the Leray-Schauder degree [7, p. 88], there exists \( x \in \Omega \) such that \( (I-T)x = 0 \). This means that there exists a solution \( x(t) \) of the PBVP (3)-(2) with \( (t, x(t), x'(t)) \in D \) for all \( t \in I \).

3. **Applications of the basic theorem.** In this section, we illustrate how Theorem 2.1 can be used to prove existence results for PBVP (1)-(2).
The first result is known (e.g., [1], [4], or [5]), but it well illustrates the power of our basic theorem.

**Theorem 3.1.** If \( f(t, x, x') \) is continuous on \( E_R = \{(t, x, x'): t \in I, \|x\| < R, \|x'\| < \infty \} \) and satisfies:

1. \( \|x'\|^2 + \langle x, f(t, x, x') \rangle > 0 \) for all \( (t, x, x') \in E_R \) provided \( \|x\| = R \) and \( \langle x, x' \rangle = 0 \);
2. \( \|f(t, x, x')\| \leq \varphi(\|x'\|) \) for all \( (t, x, x') \in E_R \) where \( \varphi \) is a positive continuous function on \([0, \infty)\) with \( \int_0^\infty \frac{\varphi(s)}{s} \, ds = +\infty \);
3. there exists \( a \geq 0, K \geq 0 \) such that
   \[ \|f(t, x, x')\| \leq 2a(\|x'\|^2 + \langle x, f(t, x, x') \rangle) + K \quad \text{for all} \quad (t, x, x') \in E_R; \]

then there exists a solution \( x(t) \) of the PBVP (1)-(2) with \( (t, x(t), x'(t)) \in E_R \).

**Proof.** Let \( \delta_M(s) \) be a continuous function on \([0, \infty)\) with \( \delta_M(s) = 1 \) on \([0, M]\) and \( \delta_M(s) = 0 \) for \( s \geq 2M \) where \( M = M(\alpha, K, R) \) is the Nagumo-Hartman bound (see Hartman [3, p. 429]).

Define
\[
F(t, x, x') = \delta_M(\|x'\|)f(t, x, x') \quad \text{on} \quad E_R, \quad \text{and} \quad F(t, x, x') = \left(\frac{R}{\|x\|}\right)F(t, R\|x\|, x') \quad \text{if} \quad \|x\| \geq R.
\]

Then \( F(t, x, x') \) is continuous and bounded on \( I \times \mathbb{R}^n \times \mathbb{R}^n \) and satisfies (8) provided \( \|x\| \geq R \) and \( \langle x, x' \rangle = 0 \), (9), and (10) for all \( (t, x, x') \in I \times \mathbb{R}^n \times \mathbb{R}^n \).

The proof will be completed by showing that there can be constructed an open bounded set \( D \subset I \times \mathbb{R}^n \times \mathbb{R}^n \) containing \( \{(t, 0, 0): t \in I\} \) such that solutions of PBVP (4)-(2) satisfy hypothesis (H) relative to \( D \).

For each \( \lambda \in [0, 1] \), let \( \tilde{F}_\lambda(t, x, x') = \lambda F(t, x, x') + (1-\lambda)F(t, x, x') \) where \( F \) is defined as above. Then for all \( \lambda \in [0, 1] \) and all \( (t, x, x') \in I \times \mathbb{R}^n \times \mathbb{R}^n \),

\[
\|x'\|^2 + \langle x, \tilde{F}_\lambda(t, x, x') \rangle > 0 \quad \text{provided} \quad \|x\| \geq R
\]

and \( \langle x, \lambda \rangle = 0 \). Let \( x(t) \) be any solution of (4)-(2). Define \( u(t) = \|x(t)\|^2 = \langle x(t), x(t) \rangle \). Because \( u(t) \) satisfies the periodic boundary conditions (2), \( u(t) \) can assume its maximum at \( t_0 \in I \) only if \( u(t_0) = 0, u'(t_0) \leq 0 \). Claim \( u(t) < R^2 \) for all \( t \in I \). Assume not; then there exists \( t_0 \in I \) at which \( u(t) \) assumes its maximum with \( u(t_0) \geq R^2, u'(t_0) = 0, \) and \( u''(t_0) \leq 0 \). But (11) implies that \( u''(t_0) > 0 \) which is a contradiction. Hence, \( \|x(t)\| < R \) for all \( t \in I \). For \( (t, x, x') \in E_R \) and \( \lambda \in [0, 1] \), \( F_\lambda(t, x, x') \) is bounded which implies that \( F_\lambda(t, x, x') \) satisfies a Nagumo-Hartman condition (conditions (9) and (10) with \( x = 0 \) and a \( K' \) in general different from \( K \) and \( \varphi(s) = K' \)). Hence, there exists a \( M' > 0 \) such that if \( x(t) \) is any solution of (4) on \( I \) with \( \|x(t)\| < R \), then \( \|x'(t)\| < M' \).
Define $D = \{(t, x, x'): t \in I, \|x\| < R, \|x'\| < M'\}$. From the observations made above it is immediate that solutions of (4)-(2) satisfy (H) relative to $D$. By Theorem 2.1, the PBVP (3)-(2) has a solution $x(t)$ with $\|x(t)\| < R$. Since $F(t, x, x')$ satisfies (9) and (10), $\|x'(t)\| < M$ on $[0, 1]$ which implies that $x(t)$ is a solution of PBVP (1)-(2) on $I$ with $(t, x(t), x'(t)) \in E_R$.

Equality can be permitted in (8) by an approximating argument like the one given in [3, p. 433].

The preceding theorem can be generalized by replacing $\|x\|^2$ by a function $V(t, x)$ which plays essentially the same role. In so doing, we obtain results similar to those obtained by Knobloch [4] and Mawhin [5].

Assume $f(\tau, x, x') : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and let $R^+$ denote the nonnegative reals.

**Definition.** Let $V \in C^2(I \times \mathbb{R}^n \times \mathbb{R}^n, R^+)$ be such that:

(a) there exists $R > 0$ such that $\Phi = \{x \in \mathbb{R}^n : V(t, x) < R, t \in I\}$ is bounded,

(b) $U(t, x, x') = V_{xx}(t, x) + 2(V_{tx}(t, x), x') + (V_{x}(t, x), x') \geq 0$,

(c) $V(t, x, x') = U(t, x, x') + (V_x(t, x), f(t, x, x')) > 0$ provided $V(t, x) = R$

and $V(t, x) + (V_x(t, x), x') = 0$,

(d) $(V_x(t, x), x') > 0$ for all $(t, x)$ such that $V(t, x) = R$,

(e) $V(0, x) = V(1, x)$, $V_t(0, x) + (V_x(0, x), x') \geq V_t(1, x) + (V_x(1, x), x')$.

Any such $V$ is called a bounding Lyapunov function relative to (1).

**Theorem 3.2.** If $V$ is a bounding Lyapunov function for (1), then for every $\lambda \in [0, 1]$ every solution $x(t)$ of the PBVP:

\[
x'' = f_\lambda(t, x, x')
\]

where $f_\lambda = \lambda f + (1 - \lambda)f$ is such that $V(\tau, x(\tau)) > R$ for some $\tau \in I$ or $V(t, x(t)) < R$ for all $t \in I$.

**Proof.** Let $x(t)$ be any solution of the PBVP (12)-(2) and let $m(t) = V(t, x(t))$, then $m'(t) = V_t(t, x(t)) + (V_x(t, x(t)), x'(t))$ and

\[
m''(t) = U(t, x(t), x'(t)) + (V_x(t, x(t)), f_\lambda(t, x(t), x'(t))).
\]

By (b), (c), and (d), $m''(t) > 0$ if $V(t, x(t)) = R$ and $V_t(t, x(t)) + (V_x(t, x(t)), x'(t)) = 0$. If there exists $\tau \in I$ such that $m(\tau) > R$, we are through.

Assume $m(t) \leq R$ for all $t \in [0, 1]$. If there exists $t_0 \in I$ such that $m(t_0) = R$, then $m'(t_0) = 0$ and $m''(t_0) \leq 0$ since $m(0) = m(1)$ and $m'(t_0) \geq m'(1)$ by (e). But this is impossible by the observation made above that $m''(t_0) > 0$. Hence, $m(t) < R$ on $I$ and the conclusion of the theorem follows.

Our next theorem is similar to Theorem 6.1 [5].
**Theorem 3.3.** If $V$ is a positive definite bounding Lyapunov function relative to (1) and if there exists $S > 0$ such that for any $\lambda \in [0, 1]$ any solution $x(t)$ of PBVP (12)-(2) with $V(t, x(t)) < R$ on $I$ satisfies $\|x'(t)\| < S$ for $t \in I$, then PBVP (1)-(2) has at least one solution $x(t)$ with $V(t, x(t)) < R$.

**Proof.** Let $D = \{(t, x, x'): t \in I, V(t, x) < R, \|x'\| < S\}$. By Theorem 3.2, solutions of (12)-(2) satisfy (H) relative to $D$. Hence, by Theorem 1.2, the conclusion follows.

There are several ways of ensuring the a priori bound condition on the derivative of solutions of (12)-(2) and hence we have the following corollaries.

**Corollary 3.4.** If $V$ is a bounding positive definite Lyapunov function for (1) and if $f(t, x, x')$ satisfies (9) and (10) for all $t \in I$, $x \in \Phi$, $\|x'\| < \infty$, then PBVP (1)-(2) has a solution $x(t) \in \Phi$ for all $t \in I$.

**Corollary 3.5.** If $V$ is a bounding positive definite Lyapunov function for (1), $f(t, x, x')$ satisfies (9) for all $t \in I$, $x \in \Phi$, $\|x'\| < \infty$, and if there exists $\beta \geq 0$, $L \geq 0$ such that

$$\|f(t, x, x')\| \leq \beta(U(t, x, x') + \langle V_x(t, x), f(t, x, x') \rangle) + L$$

for all $t \in I$, $x \in \Phi$, and $\|x'\| \leq \infty$, then PBVP (1)-(2) has a solution $x(t) \in \Phi$ for all $t \in I$.

**Corollary 3.6.** If $V$ is a bounding positive definite Lyapunov function for (1), if $f(t, x, x')$ satisfies (9) for all $t \in I$, $x \in \Phi$, $\|x'\| < \infty$, and if there exists a function $\rho(t) \in C^2(I)$ such that

$$\|f(t, x, x')\| \leq \rho''(t) \quad \text{for all } t \in I, x \in \Phi, \|x'\| < \infty,$$

then PBVP (1)-(2) has a solution $x(t) \in \Phi$ for all $t \in I$.

4. **Further consequences.** In this section, we present two further applications of Theorem 2.1. The first theorem presented shows that the bounding set $\Phi$ need not be given in terms of a bounding Lyapunov function. Assume $f(t, x, x') : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

**Theorem 4.1.** Let $G$ be a bounded convex open set in $\mathbb{R}^n$ containing 0 and assume there is a function $N : \partial G \to \mathbb{R}^n$ satisfying:

$$\langle N(x), x \rangle > 0 \quad \text{for all } x \in \partial G,$$

$$G \subseteq \{y : \langle N(x), y - x \rangle \leq 0 \quad \text{for each } x \in \partial G\},$$

$$\langle N(x), f(t, x, x') \rangle > 0 \quad \text{for all } t \in I, x \in \partial G,$$

$$x' \text{ with } \langle N(x), x' \rangle = 0,$$

then for every $\lambda \in [0, 1]$ every solution $x(t)$ of (12)-(2) is such that $x(\tau) \notin G$ for some $\tau \in I$ or $x(t) \in G$ for all $t \in I$. 


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Remark. Conditions (16) and (17) say that \( N(x) \) is an outer normal for \( G \). Gustafson and Schmitt [2] have used a similar outer normal condition to study existence of periodic solutions for delay differential equations.

Proof. Let \( x(t) \) be any solution of (12)-(2). If \( x(\tau) \notin G \) for some \( \tau \in I \), we are through so assume \( x(t) \in G \) for all \( t \in I \).

If \( x(t_0) \in \partial G \) for some \( t_0 \in I \), we may assume \( t_0 \in [0, 1) \). By (16) and (18), \( \langle N(x(t_0)), f_x(t_0, x(t_0), x'(t_0)) \rangle > 0 \) and hence there is an \( h > 0 \) such that \( \langle N(x(t_0)), x''(t) \rangle > 0 \) for all \( t \in [t_0, t_0+h] \). Since \( x(t) \in G \), \( \langle N(x(t_0)), x'(t_0) \rangle = 0 \). Looking at the Taylor expansion for \( x(t) \), we have immediately that

\[
\langle N(x(t_0)), x(t) - x(t_0) \rangle = (t - t_0)\langle N(x(t_0)), x'(t_0) \rangle + \frac{1}{2}(t - t_0)^2\langle N(x(t_0)), y(\tilde{t}) \rangle
\]

where \( y(\tilde{t}) = (x_1'(\tilde{t}), \ldots, x_n'(\tilde{t})) \) and \( t_0 < \tilde{t} < t < t_0 + h \) for all \( i = 1, \ldots, n \). From this, \( \langle N(x(t_0)), x(t) - x(t_0) \rangle > 0 \) meaning that \( x(t) \notin G \), which is a contradiction.

Our existence theorem then follows.

Theorem 4.2. If \( G \) is a bounded convex open set in \( \mathbb{R}^n \) containing 0, if there is a function \( N: \partial G \to \mathbb{R}^n \) satisfying (16), (17), and (18), and if there exists \( S > 0 \) such that for any \( \lambda \in [0, 1] \) any solution \( x(t) \) of PBVP (12)-(2) with \( x(t) \in G \) for all \( t \in I \) satisfies \( \|x'(t)\| < S \) for \( t \in I \), then PBVP (1)-(2) has at least one solution with \( x(t) \in G \) for all \( t \in I \).

Proof. Let \( D = \{(t, x, x'): t \in I, x \in G, \|x'\| < S\} \). By Theorem 4.1 solutions of (11)-(2) satisfy (H) relative to \( D \). Result then follows immediately from Theorem 2.1.

Remark. One can state corollaries of the above theorem analogous to Corollaries 3.4, 3.5, and 3.6.

In \( \mathbb{R}^n \), let \( x \leq y \) if and only if \( x_i \leq y_i, 1 \leq i \leq n \), and \( x < y \) if and only if \( x_i < y_i, 1 \leq i \leq n \).

Let \( f(t, x, x') \) be continuous on \( \{(t, x, x'): t \in I, \alpha(t) \leq x \leq \beta(t), x' \in \mathbb{R}^n\} \) where \( \alpha, \beta: I \to \mathbb{R}^n \), \( \alpha(t) < \beta(t) \) are twice continuously differentiable with

\[
\alpha(0) = \beta(0) = \alpha'(1), \quad \alpha'(0) \leq \alpha'(1), \quad \beta(0) \leq \beta'(1).
\]

Assume also that \( \alpha, \beta \) are strict lower, upper solutions of (1), i.e.,

\[
\alpha''(t) > f(t, x_1, \ldots, x_{i-1}, \alpha_i(t), x_{i+1}, \ldots, x_n, x'_i), \quad x'_{i-1}, x'_i(t), x'_{i+1}, \ldots, x_n, x'_i,
\]

\[
\beta''(t) < f(t, x_1, \ldots, x_{i-1}, \beta_i(t), x_{i+1}, \ldots, x_n, x'_i), \quad x'_{i-1}, \beta'_i(t), x'_{i+1}, \ldots, x_n.
\]
and

\( \alpha_i''(t) > \alpha_i(t), \quad \beta_i''(t) < \beta_i(t) \)

for \( \alpha_i(t) \leq x_j \leq \beta_j(t), \ j \neq i, \ i = 1, \ldots, n. \)

We now can state our final result.

**Theorem 4.3.** If \( f \) is continuous on \( \{ (t, x, x') : t \in I, \ \alpha(t) \leq x \leq \beta(t), \ x' \in \mathbb{R}^n \} \) where \( \alpha, \beta \) are strict periodic lower, upper solutions of (1) satisfying (19), (20), and (21), and if there exists \( S > 0 \) such that for any \( \lambda \in [0, 1] \) any solution \( x(t) \) of (12)-(2) with \( \alpha(t) \leq x(t) \leq \beta(t) \) on \( I \) satisfies \( \| x'(t) \| < S \) then PBVP (1)-(2) has a solution \( x(t) \) with \( \alpha(t) < x(t) < \beta(t) \).

The proof is similar to those previously given and is for this reason omitted. By a proper modification of \( f(t, x, x') \), condition (21) can be dropped and equality can be permitted in (20). With that observation, we have a generalization of Theorem 4.1 in [1].

**References**


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