THE VOLUME OF A REGION DEFINED BY POLYNOMIAL INEQUALITIES

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Abstract. Let $P(x)$ be a polynomial on $\mathbb{R}^n$ with nonnegative coefficients. We develop a simple necessary and sufficient condition that the set $S=\{x \in \mathbb{R}^n | x_i \geq 0, P(x) \leq 1\}$ shall have finite volume. A corresponding result where $P(x)$ is replaced by a collection of polynomials is an easy corollary. Finally, the necessary and sufficient conditions for the special case that $P$ is a product of linear forms is also given.

Let $P(x)$ be a polynomial on $\mathbb{R}^n$ with nonnegative coefficients, and without constant term (to avoid trivial complications).

$$P(x) = \sum_{r=1}^{k} r_v x_1^{c_v(1)} x_2^{c_v(2)} \cdots x_n^{c_v(n)}, \quad r_v > 0.$$  

The vectors $c_v = (c_v(1), c_v(2), \cdots, c_v(n))$ are called the exponents of $P$. Let $C$ be the closed convex cone in $\mathbb{R}^n$ generated by the $c_v$, i.e., the elements of $C$ are all linear combinations $p_1 c_1 + p_2 c_2 + \cdots + p_k c_k$ with $p_i \geq 0$. Let $\langle \cdot, \cdot \rangle$ be the usual inner product in $\mathbb{R}^n$, and let $C^*$ be the dual cone to $C$ with respect to this scalar product; i.e., $C^*$ is the set of $y \in \mathbb{R}^n$ such that $\langle y, x \rangle \geq 0 \ \forall x \in C$. Note that $C^*$ contains the first $2^n$-gant in $\mathbb{R}^n$, so $C^*$ has nonempty interior.

There are several well-known features of the above situation, which it is easy to establish using separation properties of convex sets. Thus if $b$ is not an interior point of $C$, there exists $d \in C^*$, $d \neq 0$, such that $\langle d, b \rangle \leq 0$. While if $b \neq 0$ is an interior point of $C$, then there exists a positive constant $p$ such that $\langle d, b \rangle \geq p \langle d, d \rangle^{1/2} \ \forall d \in C^*$, as an easy compactness argument shows. Then we have

**Theorem 1.** The set $S=\{x | x_i \geq 0, P(x) \leq 1\}$ is of finite volume if and only if the vector $m=(1, 1, \cdots, 1)$ is an interior point of $C$. (In particular, $C$ must have a nonempty interior.)
Proof. \( (S \) is convex, but we do not need this fact.\)

\[
\text{Vol } S = \int_{x, x^T P(x) \leq 1} dx = \int_{P(e^{-u}) \leq 1} e^{-\langle m, u \rangle} \, du.
\]

Now pick a vector \( y \) such that \( \langle e_v, y \rangle \geq \log kr_e \). Then if \( u \in C^* + y \),

\[
P(e^{-u}) = \sum_{v=1}^{k} r_v e^{-\langle e_v, u \rangle} \leq \sum_{v} r_v \frac{1}{kr_v} = 1.
\]

So \( \{ u \in \mathbb{R}^n | P(e^{-u}) \leq 1 \} \subset C^* + y \).

Also pick a vector \( w \) such that \( \langle e_v, w \rangle \leq \log r_v \). Then if \( P(e^{-u}) \leq 1 \), we must have \( r_v e^{-\langle e_v, u \rangle} \leq 1 \), which implies that \( \langle e_v, u \rangle \geq \log r_v \), which implies that \( \langle e_v, u - w \rangle \geq 0 \), i.e., \( u \in w + C^* \).

Thus the set \( \{ u \in \mathbb{R}^n | P(e^{-u}) \leq 1 \} \) is contained in some translate of \( C^* \), and contains a second translate. It follows that \( \text{Vol } S \) is finite if and only if \( \int_{C^*} e^{-\langle m, u \rangle} \, du \) is finite. But if \( m \) is an interior point of \( C \), then \( \langle m, u \rangle \geq p \langle u, u \rangle^{1/2} \) for \( u \in C^* \), and the integral is obviously finite. While if \( m \) is not an interior point, it is easy to see that the above integral diverges, completing the proof.

Corollary. Let \( P_1, P_2, \cdots, P_r \) be polynomials on \( \mathbb{R}^n \) with nonnegative coefficients. The set

\[
S = \{ x \mid x_i \geq 0, P_1(x) \leq 1, P_2(x) \leq 1, \cdots, P_r(x) \leq 1 \}
\]

is of finite volume if and only if \( m = (1, 1, \cdots, 1) \) is an interior point of the cone generated by the exponents of all the polynomials \( P_i \).

For if \( x \in S \), then \( r^{-1}P_1(x) + r^{-1}P_2(x) + \cdots + r^{-1}P_r(x) \leq 1 \), while if \( P_1(x) + P_2(x) + \cdots + P_r(x) \leq 1 \), \( x \in S \).

Next, we apply the above theorem to the case when \( P(x) \) is a product of linear forms on \( \mathbb{R}^n \).

\[
P(x) = \prod_{i=1}^{k} (a_{v(1)}x_1 + a_{v(2)}x_2 + \cdots + a_{v(n)}x_n),
\]

each linear form having nonnegative coefficients not all zero. Let \( U \) be a subset of \( \{ 1, 2, \cdots, n \} \). We say that the support of the linear form \( a_1x_1 + a_2x_2 + \cdots + a_nx_n \) is \( U \) if \( a_i \neq 0 \) for \( i \in U \), and \( a_i = 0 \) for \( i \notin U \). For any subset \( U \), let \( N(U) \) be the number of linear forms in product for \( P(x) \) whose support is contained in \( U \). Then we have:

Theorem 2. \( \text{Vol } S \) is finite if and only if for every proper subset \( U \), we have \( N(U)/\text{card } U < k/n \).
To prove the "if" part, let $u=(u_1, u_2, \cdots, u_n) \in C^*$, and suppose without loss of generality that $u_1 \geq u_2 \geq \cdots \geq u_n$. For $1 \leq r \leq n$, put $N_r = N(\{1, 2, \cdots, r\})$. Then the vector $c = (N_1, N_2 - N_1, N_3 - N_2, \cdots, N_n - N_{n-1})$ is an exponent of $P$.

Hence

$$\langle c, u \rangle = N_1(u_1 - u_2) + N_2(u_2 - u_3) + \cdots + N_{n-1}(u_{n-1} - u_n) + k u_n \leq (k/n)(u_1 + u_2 + \cdots + u_n)$$

with equality if and only if $u_1 = u_2 = \cdots = u_n$. Since $\langle c, u \rangle \geq 0$, we obtain $\langle m, u \rangle > 0$ if the components of $u$ are not all equal. While if the components of $u$ are all equal and not all zero, then since $u \in C^*$, the components of $u$ are all positive, and again $\langle m, u \rangle > 0$. This proves that $m$ is an interior point of $C$, and completes the proof of "if".

For the "only if" part, suppose that, for $U = \{1, 2, \cdots, r\}$, $N(U)/r \leq k/n$. We will show $m$ cannot be an interior part of $C$. Consider the vector $u$ whose first $r$ components are equal to $n-r$, and whose remaining $n-r$ components are equal to $-r$. For any exponent $c = (c_1, c_2, \cdots, c_n)$, we have

$$\langle c, u \rangle = (c_1 + c_2 + \cdots + c_r)(n-r) - (c_{r+1} + \cdots + c_r)r = (c_1 + c_2 + \cdots + c_r)n - kr.$$ 

As $c$ runs through all exponents of $P$, $\langle c, u \rangle$ will be minimum when $c_1 + c_2 + \cdots + c_r$ is as small as possible, i.e., when $c_1 + c_2 + \cdots + c_r = N(U)$. Since $N(U) \geq kr/n$, we have always $\langle c, u \rangle \geq 0$ for any exponent $c$. Hence $u \in C^*$; but $\langle m, u \rangle = 0$ and this proves $m$ is not an interior point of $C$, and completes the proof.

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