A COUNTABLY DISTRIBUTIVE COMPLETE BOOLEAN ALGEBRA NOT UNCOUNTABLY REPRESENTABLE

JOHN GREGORY

Abstract. It is proved from the Continuum Hypothesis that there exists an ω-distributive complete Boolean algebra which is not ω₁-representable.

Karp [1, Corollary to main theorem] shows that there is an ω-distributive non-\(|2^ω|\)-representable \(2^ω\)-complete Boolean algebra. Karp later pointed out that, for every cardinal \(K \geq |2^ω|\), there is an ω-distributive non-\(|2^ω|\)-representable \(K\)-complete Boolean algebra. The following strengthens the completeness requirement.

Theorem 1. The Continuum Hypothesis implies the existence of an ω-distributive non-ω₁-representable complete Boolean algebra.

In [3, Definition 1.2], the property \(P_K\) was defined for cardinals \(K\). We will prove the following theorem.

Theorem 2. If \(|2^ω| < |2^{ω₁}|\), then there exists an ω-distributive complete Boolean algebra that does not have the property \(P_{ω₁}\).

Theorem 1 is a consequence of Theorem 2, since the Continuum Hypothesis implies both \(|2^ω| < |2^{ω₁}|\) and (by [3, Theorem 4.2]) the equivalence of ω₁-representability with property \(P_{ω₁}\).

Assuming the negation of Souslin’s Hypothesis, Smith’s [3, Theorem 3.4] implies our theorems. However, R. Jensen has shown that Souslin’s Hypothesis is consistent with the Continuum Hypothesis; so Smith’s result does not imply ours.

1. Preliminaries. Let \(ω\) be the first infinite cardinal; \(ω₁\) is the first uncountable cardinal; \(x^y\) is the set of all functions from \(y\) into \(x\); \(|x|\) is the cardinal of \(x\) (cardinals are initial ordinals); \(Dm(x)\) is the domain of function \(x\); \(x|y\) is the restriction of a function \(x\) to the domain \(Dm(x) \cap y\). We assume the axiom of choice.

Received by the editors November 15, 1972 and, in revised form, February 7, 1973.

AMS (MOS) subject classifications (1970). Primary 06A40; Secondary 04A30.

Key words and phrases. Complete Boolean algebra, ω-distributive Boolean algebra, ω₁-representable Boolean algebra, Continuum Hypothesis.

1 This research was partially supported by a National Science Foundation Grant of Carol Karp, NSF GP 34033.
A COUNTABLY DISTRIBUTIVE COMPLETE BOOLEAN ALGEBRA

The Boolean algebraic definitions are essentially those of [2]. A Boolean algebra $C$ is ordered by: $a \leq b$ iff the infimum $a \land b = a$. $C$ is $K$-complete if the infimum (or meet) $\land S$ exists for each set $S$ of no more than $K$ many elements of $C$. Then the supremum (or join) $\lor S$ exists for all such $S$. $C$ is complete if the infimum of every set of elements exists. A $K$-complete Boolean algebra $C$ is $K$-distributive iff

$$
\bigwedge_{a \leq K} \bigvee_{\beta < K} A(a, \beta) \leq \bigvee_{H \leq K} \bigwedge_{a \leq K} A(a, H(a))
$$

for all functions $A$ into $C$. A Boolean algebra is $K$-representable iff it is isomorphic to the quotient of a $K$-field of sets by a $K$-ideal. Smith defined the following distributive-like property $P_K$ which implies $K$-representability.

**Definition.** A Boolean algebra $C$ has the property $P_K$ iff: if $A$ is a function from $K \times K$ into $C$ and if $\bigwedge_{a \leq K} \bigvee_{\beta < K} A(a, \beta)$ exists and is not zero, then there is a function $H$ from $K$ into $K$ such that, for all $\gamma < K$, either $\bigwedge_{a \leq \gamma} A(a, H(a))$ does not exist or else is not zero.

A subset $S$ of $C$ is dense iff $S$ does not contain the zero of $C$ and, for all nonzero elements $b$ of $C$, there exists $x \in S$ such that $x \leq b$.

The following is well known and is related to "forcing".

A set $S$ ordered by $\leq$ satisfies

(1) $$
\forall x \leq y, \exists z \leq x \text{ such that, for all } w, \text{ not both } w \leq y \text{ and } w \leq z
$$

if and only if there exists an (algebraically unique) complete Boolean algebra $C$ such that $S$ is a dense subset of $C$ and the ordering of $C$ extends that of $S$. (See [2, Example 12(B)] and [2, §35].)

If $S$ is dense in $C$ and $x, y, z$ range over $S$, then: $x \leq \bigwedge X$ iff, for all $b \in X$, $x \leq b$; $x \leq \text{complement of } b$ iff, for all $z$, not both $z \leq x$ and $z \leq b$; $x \leq \bigvee X$ iff, for all $y \leq x$, there is $z \leq y$ and $b \in X$ such that $z \leq b$.

2. **Proof of Theorem 2.** Assume $\Omega = |2^\omega| < |2^{\omega^n}|$. Suppose there did not exist an $\omega$-distributive complete Boolean algebra not having property $P_\omega$. It will suffice to reach a contradiction. Let $X$ be the set of all nonempty countable sequences $f$ into $2^\omega$ (i.e., of all functions $f$ whose nonempty domain is a countable ordinal and whose range is a subset of $2^\omega$).

Define $\mathcal{F}$ to be the set of all $F$ such that: $F$ is a function from a subset of $X$ into $X$; if $F(f)$ defined, then $\text{Dm}(F(f)) = \text{Dm}(f) + 1$; if $F(f)$ defined, then so is $F(f\upharpoonright \alpha) = F(f)(\alpha + 1)$ whenever $0 < \alpha \leq \text{Dm}(f)$.

There exists a $\Omega$-sequence $E$ that enumerates the set of all ordered pairs $(f, t)$ such that $f \in X$ and countable $t \leq X$, each such pair occurring $\Omega$ times in the sequence $E$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
By induction on $\alpha \leq \Omega$, $F_\alpha$ will now be constructed such that $F_\alpha \in \mathcal{F}$, $|F_\alpha| \leq |\omega \cup \alpha|$, and $F_\alpha$ extends $F_\beta$ whenever $\alpha < \beta \leq \Omega$.

For limit ordinal $\lambda \leq \Omega$, let $F_\lambda$ be the union of all $F_\alpha$, $\alpha < \lambda$. (In particular, $F_0$ is empty.)

Given $F = F_\alpha$, $\alpha \leq \Omega$, $G = F_{\alpha+1}$ is defined by cases as follows. In each case below, note that $G \in \mathcal{F}$, $G$ extends $F$, and $G$ has only countably more elements than $F$. (Cases 1, 2, 3 are for Lemmas 1(1), 1(2), 2 respectively.)

**Case 1.** $E(\alpha)$ is some $(f, 0)$ such that $\text{Dm}(f) = 2$. Since $(2^{\omega})^1$ has cardinality greater than $F_\alpha$, there is $h \in (2^{\omega})^1$ such that $F(h)$ is not defined. Extend $F$ to $G$ by: $G(h) = f$; $G(k) = F(k)$ whenever defined.

**Case 2.** $E(\alpha)$ is some $(f, \{g\})$ such that $f$ has a domain $\gamma + 1$, $g$ has a domain $\delta < \gamma$, and $F(g) = f(\delta + 1)$. Then there is $h \in (2^{\omega})^1$ such that $h$ extends $g$ and $F(h\upharpoonright(\delta + 1))$ is not defined. Then $F(h\upharpoonright\beta)$ is defined iff $0 < \beta \leq \delta$; for such $\beta$, $h\upharpoonright\beta = g\upharpoonright\beta$. Extend $F$ to $G$ by: $G(h\upharpoonright\beta) = f(\beta + 1)$ whenever $\delta < \beta \leq \gamma$; $G(k) = F(k)$ whenever defined.

**Case 3.** $E(\alpha)$ is some $(f, t)$ such that: $f$ has a domain $\gamma + 1$; for every $k \in t$, $F(k) \subseteq f$ is defined; there are $\Omega$ many $h \in (2^{\omega})^1$ such that $h$ is a union of elements of $t$. Then there exists $h \in (2^{\omega})^1$ such that $h$ is a union of elements of $t$ but $F(h)$ is not defined. Extend $F$ to $G$ by: $G(h) = f$; $G(k) = F(k)$ whenever defined.

**Case 4.** The first three cases do not hold. Let $G$ be $F$.

This finishes the construction of $F_\alpha$, $\alpha \leq \Omega$. From now on, let $F$ be $F_\Omega$.

Given $B \in (2^{\omega})^\omega$, define $T$ to be \{ $f$ $|$ $F(f) \subseteq B$ $\}$.

$T$ will turn out to be a dense subset of a complete $\omega$-distributive Boolean algebra. But first some facts about $T$ will be proved.

**Lemma 1.** (1) $T$ is nonempty. (2) If $g$ is in both $T$ and $(2^{\omega})^\delta$ and if $\delta < \gamma < \omega_1$, then $g$ has at least two distinct extensions in $T \cap (2^{\omega})^\gamma$. (3) If $f \in T$ and nonzero $\beta \leq \text{Dm}(f)$, then $f\upharpoonright\beta \in T$.

**Proof.** (1) For some $\alpha$, $E(\alpha)$ is $(B\upharpoonright2, 0)$. By Case 1 of the construction of $F_{\alpha+1}$, there is $h$ such that $F_{\alpha+1}(h) = B\upharpoonright2$. Then $F(h) = B\upharpoonright2$ and this $h$ is in $T$.

(2) Then for some $\beta < \Omega$, $F_{\alpha}(g) = B\upharpoonright(\delta + 1)$ is defined. For $\Omega$ many $\alpha > \beta$, $E(\alpha)$ is $(B\upharpoonright(\gamma + 1), \{g\})$. For each such $\alpha$, by Case 2 of the construction of $F_{\alpha+1}$, there is assigned $h \in (2^{\omega})^\gamma$ such that $h$ extends $g$, $F_{\alpha+1}(h) = B\upharpoonright(\gamma + 1)$, and $F_\alpha(h)$ is not defined. Then $h$ is in $T \cap (2^{\omega})^\gamma$ and $h$ extends $g$. Since $F_\alpha(h)$ is not defined, $h$ was not assigned to a smaller $\alpha$; thus, distinct such $\alpha$'s give distinct $h$'s.

(3) $f\upharpoonright\beta \in B$. Thus $F(f)$ is defined. Since $F \in \mathcal{F}$, $F(f\upharpoonright\beta) \subseteq F(f) \subseteq B$. Thus, $f\upharpoonright\beta \in T$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Lemma 2. If countable \( t \subseteq T \) and there are \( \Omega \) many \( h \in (2^\omega)' \) such that \( h \) is a union of elements of \( t \), then there exists \( h \in T \cap (2^\omega)' \) such that \( h \) is a union of elements of \( t \).

Proof. For each \( k \in t \), \( F(k) \subseteq B \) and some \( F_k(k) \), \( \beta < \Omega \), is defined. Since \( \Omega \) is not cofinal with \( \omega \), there exists \( \beta < \Omega \) such that, for all \( k \in t \), \( F_k(k) \) is defined. For some \( \alpha > \beta \), \( E(\alpha) \) is \( (B^\uparrow(\gamma + 1), t) \). Then Case 3 of the definition of \( F_{n+1} \) gives \( h \in (2^\omega)' \) such that \( h \) is a union of elements of \( t \) and \( F_{n+1}(h) = B^\uparrow(\gamma + 1) \). Then \( F(h) \subseteq B \) and \( h \in T \).

Order \( T \) by: \( f \leq g \) iff \( f \) extends \( g \). Then \( T \) satisfies (1) of §1. So there is a (algebraically unique) complete Boolean algebra \( C \) such that \( T \) is a dense subset of \( C \) and the ordering of \( C \) extends that of \( T \).

Lemma 3. \( C \) is an \( \omega \)-distributive Boolean algebra.

Proof. Suppose that \( A \) is a function on \( \omega \times K \) into \( C \). By the definition of \( \omega \)-distributivity and by the fact that \( T \) is dense in \( C \), it suffices to show for every \( f \in T \) that: if \( \forall \alpha \leq m \; \forall x \in A(m, \alpha) \), then there is \( h \in T \) such that \( h \leq f \) and, for all \( m < \omega \), \( h \leq A(m, \alpha) \) for some \( H(m) = \alpha < K \).

Define \( P(r) \in T \) for \( r \in 2^m \) by the following induction on \( m < \omega \). Let \( P(0) = f \). Suppose \( P(r) \leq f \) defined for all \( r \in 2^m \). Let \( \beta(m) \) be the least upper bound of \( \{Dm(P(r)) \mid r \in 2^m \} \). Consider any \( r \in 2^m \). Let \( u \) and \( v \) be the two extensions in \( 2^{m+1} \) of \( r \). Since \( P(r) \leq f \leq \bigvee \alpha \; A(m, \alpha) \), there exists \( g \in T \) such that \( g \leq P(r) \) and \( g \leq A(m, \alpha) \) for some \( \alpha \). By Lemma 1(2), there exist distinct extensions \( P(u) \) and \( P(v) \) of \( g \) such that they both have the same domain greater than \( \beta(m) \).

Put \( t = \{P(r) \mid r \in 2^m \} \) for some \( m < \omega \). Let \( \gamma = \bigcup \beta(m) \). By Lemma 2, there exists \( h \in T \cap (2^\omega)' \) such that \( h \) is a union of some elements of \( t \). Then \( h \leq P(0) = f \). Consider any \( m < \omega \). There is \( r \in 2^{m+1} \) such that \( P(r) \subseteq h \). There is \( \alpha < K \) such that \( h \leq P(r) \leq A(m, \alpha) \).

Lemma 4. \( T \) has an uncountable chain.

Proof. Define \( A \) on \( \omega_1 \times 2 \) into \( C \) by

\[
A(\omega \beta + m, i) = \bigvee \{f \in T \mid f(\beta)(m) = i, \; \beta < \omega_1, \; m < \omega, \; i < 2\}.
\]

By Lemma 1(2), for every \( g \in T \) there is \( f \leq g \) such that \( \beta \in Dm(f) \) and hence \( f \leq A(\omega \beta + m, f(\beta)(m)) \). Thus since \( T \) is dense, it follows that each \( \bigvee_i \; A(\omega \beta + m, i) \) is the identity of \( C \). Since \( C \) is a complete \( \omega \)-distributive Boolean algebra, it must have property \( P_\omega \) (by our earlier supposition, which is used here only). By \( P_\omega \), there exists a function \( H \) on \( \omega_1 \) into \( 2 \) such that, for all \( \gamma < \omega_1 \), the infimum \( \bigwedge_{\beta < \gamma} \; A(\beta, H(\beta)) \) is nonzero. Let \( J = \{f \in X \mid f(\beta)(m) = H(\omega \beta + m) \} \) for all \( \beta < Dm(f) \) and \( m < \omega \). Then \( J \) is an uncountable chain of functions. It now suffices to prove \( J \subseteq T \). Consider any \( f \in J \). Let \( \alpha = Dm(f) \) and \( \gamma = \omega \alpha \). Since \( T \) is dense, there exists \( g \in T \),
Then \( g(\beta)(m) = H(\omega \beta + m) \) for all \( \beta < \alpha \), \( m < \omega \); so \( g(\alpha) \) is \( f \). By Lemma 1(3), \( f \in T \).

For each \( B \in (2^\omega)^{\omega_1} \), we now choose \( T(n, B) \), \( J(n, B) \), by induction on \( n < \omega \), such that \( T(n, B) \subseteq X \), \( J(n, B) \in (2^\omega)^{\omega_1} \), and \( J(n, B) \upharpoonright \alpha \in T(n, B) \) for all nonzero \( \alpha < \omega_1 \).

\[
T(0, B) = X, \quad J(0, B) = B.
\]

Suppose \( T(n, B) \) and \( J(n, B) \) are defined. Define \( T(n+1, B) = \{ f | F(f) \subseteq J(n, B) \} \) (i.e., define \( T(n+1, B) \) from \( J(n, B) \) as \( T \) was earlier defined from \( B \)). By Lemma 4, \( T(n+1, B) \) has an uncountable branch. The union of such a branch is some element \( J(n+1, B) \) of \((2^\omega)^{\omega_1}\).

**Lemma 5.** If \( B \) and \( B' \) are such that, for all \( n < \omega \), \( J(n, B) \upharpoonright 1 \) equals \( J(n, B') \upharpoonright 1 \), then \( B \) equals \( B' \).

**Proof.** Let \( B \) and \( B' \) be such functions. It suffices to prove, by induction on nonzero \( \beta \leq \omega_1 \), that \( R(\beta): J(n, B) \upharpoonright B = J(n, B') \upharpoonright B \) for all \( n < \omega \). For \( \beta = 1 \), \( R(\beta) \) is given. For nonzero limit \( \beta \), \( R(\beta) \) follows easily from \( R(\alpha) \) for all \( \alpha < \beta \). Suppose \( R(\beta) \) holds for nonzero \( \beta \), to show \( R(\beta+1) \).

Consider any \( n < \omega \). Let \( f \) be \( J(n+1, B) \upharpoonright \beta \). By \( R(\beta) \), \( f \) equals \( J(n+1, B') \upharpoonright \beta \). Then \( F(f) \subseteq J(n, B) \) since \( f \in T(n+1, B) \); \( F(f) \subseteq J(n, B') \) since \( f \in T(n+1, B') \). Since \( F(f) \) has domain \( \beta+1 \), it follows that \( J(n, B) \upharpoonright (\beta+1) = F(f) = J(n, B') \upharpoonright (\beta+1) \).

By Lemma 5, distinct \( B \in (2^\omega)^{\omega_1} \) are assigned distinct sequences \((J(n, B) \upharpoonright 1)_{n < \omega}) \). Thus, \(|(2^\omega)^{\omega_1}| \leq |((2^\omega)^{\omega_1})^\omega| \). This contradicts the assumption that \(|2^\omega| > |2^\omega|\). Theorem 2 is proved.

It follows from the proof of Theorem 2 that: if \(|2^\omega| < |2^\omega|\), then, for some \( B \), the corresponding \( T \) defined above is a dense subset of an \( \omega \)-distributive complete Boolean algebra not having property \( P_{\omega_1} \).

**References**


4517 Jonwall Court, Columbia, South Carolina 29206

Current address: Department of Mathematics, State University of New York at Buffalo, Amherst, New York 14226