PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 42, Number 1, January 1974

A COUNTABLY DISTRIBUTIVE COMPLETE BOOLEAN ALGEBRA NOT UNCOUNTABLY REPRESENTABLE

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ABSTRACT. It is proved from the Continuum Hypothesis that there exists an ω-distributive complete Boolean algebra which is not ω_1-representable.

Karp [**1**, Corollary to main theorem] shows that there is an ω-distributive non-$|2^\omega|$-representable $|2^\omega|$-complete Boolean algebra. Karp later pointed out that, for every cardinal $K \geqq |2^\omega|$, there is an ω-distributive non-$|2^\omega|$-representable K-complete Boolean algebra. The following strengthens the completeness requirement.

THEOREM 1. *The Continuum Hypothesis implies the existence of an ω-distributive non-ω_1-representable complete Boolean algebra.*

In [**3**, Definition 1.2], the property P_K was defined for cardinals K. We will prove the following theorem.

THEOREM 2. *If $|2^\omega| < |2^{\omega_1}|$, then there exists an ω-distributive complete Boolean algebra that does not have the property P_{ω_1}.*

Theorem 1 is a consequence of Theorem 2, since the Continuum Hypothesis implies both $|2^\omega| < |2^{\omega_1}|$ and (by [**3**, Theorem 4.2]) the equivalence of ω_1-representability with property P_{ω_1}.

Assuming the negation of Souslin's Hypothesis, Smith's [**3**, Theorem 3.4] implies our theorems. However, R. Jensen has shown that Souslin's Hypothesis is consistent with the Continuum Hypothesis; so Smith's result does not imply ours.

1. **Preliminaries.** Let ω be the first infinite cardinal; ω_1 is the first uncountable cardinal; x^y is the set of all functions from y into x; $|x|$ is the cardinal of x (cardinals are initial ordinals); $\mathrm{Dm}(x)$ is the domain of function x; $x \upharpoonright y$ is the restriction of a function x to the domain $\mathrm{Dm}(x) \cap y$. We assume the axiom of choice.

Received by the editors November 15, 1972 and, in revised form, February 7, 1973.
AMS (MOS) subject classifications (1970). Primary 06A40; Secondary 04A30.
Key words and phrases. Complete Boolean algebra, ω-distributive Boolean algebra, ω_1-representable Boolean algebra, Continuum Hypothesis.
[1] This research was partially supported by a National Science Foundation Grant of Carol Karp, NSF GP 34033.

The Boolean algebraic definitions are essentially those of [2]. A Boolean algebra C is ordered by: $a \leqq b$ iff the infimum $a \wedge b = a$. C is K-complete if the infimum (or meet) $\bigwedge S$ exists for each set S of no more than K many elements of C. Then the supremum (or join) $\bigvee S$ exists for all such S. C is complete if the infimum of every set of elements exists. A K-complete Boolean algebra C is K-distributive iff

$$\bigwedge_{\alpha<K} \bigvee_{\beta<K} A(\alpha, \beta) \leqq \bigvee_{H \in K^K} \bigwedge_{\alpha<K} A(\alpha, H(\alpha))$$

for all functions A into C. A Boolean algebra is K-representable iff it is isomorphic to the quotient of a K-field of sets by a K-ideal. Smith defined the following distributive-like property P_K which implies K-representability.

DEFINITION. A Boolean algebra C has the property P_K iff: if A is a function from $K \times K$ into C and if $\bigwedge_{\alpha<K} \bigvee_{\beta<K} A(\alpha, \beta)$ exists and is not zero, then there is a function H from K into K such that, for all $\gamma < K$, either $\bigwedge_{\alpha<\gamma} A(\alpha, H(\alpha))$ does not exist or else is not zero.

A subset S of C is dense iff S does not contain the zero of C and, for all nonzero elements b of C, there exists $x \in S$ such that $x \leqq b$.

The following is well known and is related to "forcing".

A set S ordered by $\leqq$ satisfies

(1) for all $x \nleqq y$, there exists $z \leqq x$ such that, for all w, not both $w \leqq y$ and $w \leqq z$

if and only if there exists an (algebraically unique) complete Boolean algebra C such that S is a dense subset of C and the ordering of C extends that of S. (See [2, Example 12(B)] and [2, §35].)

If S is dense in C and x, y, z range over S, then: $x \leqq \bigwedge X$ iff, for all $b \in X$, $x \leqq b$; $x \leqq$complement of b iff, for all z, not both $z \leqq x$ and $z \leqq b$; $x \leqq \bigvee X$ iff, for all $y \leqq x$, there is $z \leqq y$ and $b \in X$ such that $z \leqq b$.

2. **Proof of Theorem 2.** Assume $\Omega = |2^\omega| < |2^{\omega_1}|$. Suppose there did not exist an ω-distributive complete Boolean algebra not having property P_{ω_1}. It will suffice to reach a contradiction. Let X be the set of all nonempty countable sequences f into 2^ω (i.e., of all functions f whose nonempty domain is a countable ordinal and whose range is a subset of 2^ω).

Define $\mathscr{F}$ to be the set of all F such that: F is a function from a subset of X into X; if $F(f)$ defined, then $\mathrm{Dm}(F(f)) = \mathrm{Dm}(f) + 1$; if $F(f)$ defined, then so is $F(f \upharpoonright \alpha) = F(f) \upharpoonright (\alpha + 1)$ whenever $0 < \alpha \leqq \mathrm{Dm}(f)$.

There exists a Ω-sequence E that enumerates the set of all ordered pairs (f, t) such that $f \in X$ and countable $t \subseteq X$, each such pair occurring Ω times in the sequence E.

By induction on $\alpha \leqq \Omega$, F_α will now be constructed such that $F_\alpha \in \mathscr{F}$, $|F_\alpha| \leqq |\omega \cup \alpha|$, and F_β extends F_α whenever $\alpha < \beta \leqq \Omega$.

For limit ordinal $\lambda \leqq \Omega$, let F_λ be the union of all F_α, $\alpha < \lambda$. (In particular, F_0 is empty.)

Given $F = F_\alpha$, $\alpha < \Omega$, $G = F_{\alpha+1}$ is defined by cases as follows. In each case below, note that $G \in \mathscr{F}$, G extends F, and G has only countably more elements than F. (Cases 1, 2, 3 are for Lemmas 1(1), 1(2), 2 respectively.)

Case 1. $E(\alpha)$ is some $(f, 0)$ such that $\mathrm{Dm}(f) = 2$. Since $(2^\omega)^1$ has cardinality greater than F_α, there is $h \in (2^\omega)^1$ such that $F(h)$ is not defined. Extend F to G by: $G(h) = f$; $G(k) = F(k)$ whenever defined.

Case 2. $E(\alpha)$ is some $(f, \{g\})$ such that f has a domain $\gamma+1$, g has a domain $\delta < \gamma$, and $F(g) = f \upharpoonright (\delta+1)$. Then there is $h \in (2^\omega)^\gamma$ such that h extends g and $F(h \upharpoonright (\delta+1))$ is not defined. Then $F(h \upharpoonright \beta)$ is defined iff $0 < \beta \leqq \delta$; for such β, $h \upharpoonright \beta$ is $g \upharpoonright \beta$. Extend F to G by: $G(h \upharpoonright \beta) = f \upharpoonright (\beta+1)$ whenever $\delta < \beta \leqq \gamma$; $G(k) = F(k)$ whenever defined.

Case 3. $E(\alpha)$ is some (f, t) such that: f has a domain $\gamma+1$; for every $k \in t$, $F(k) \subset f$ is defined; there are Ω many $h \in (2^\omega)^\gamma$ such that h is a union of elements of t. Then there exists $h \in (2^\omega)^\gamma$ such that h is a union of elements of t but $F(h)$ is not defined. Extend F to G by: $G(h) = f$; $G(k) = F(k)$ whenever defined.

Case 4. The first three cases do not hold. Let G be F.

This finishes the construction of F_α, $\alpha \leqq \Omega$. From now on, let F be F_Ω.

Given $B \in (2^\omega)^{\omega_1}$, define T to be $\{f \mid F(f) \subset B\}$.

T will turn out to be a dense subset of a complete ω-distributive Boolean algebra. But first some facts about T will be proved.

LEMMA 1. (1) *T is nonempty.* (2) *If g is in both T and $(2^\omega)^\delta$ and if $\delta < \gamma < \omega_1$, then g has at least two distinct extensions in $T \cap (2^\omega)^\gamma$.* (3) *If $f \in T$ and nonzero $\beta \leqq \mathrm{Dm}(f)$, then $f \upharpoonright \beta \in T$.*

PROOF. (1) For some α, $E(\alpha)$ is $(B \upharpoonright 2, 0)$. By Case 1 of the construction of $F_{\alpha+1}$, there is h such that $F_{\alpha+1}(h) = B \upharpoonright 2$. Then $F(h) = B \upharpoonright 2$ and this h is in T.

(2) Then for some $\beta < \Omega$, $F_\beta(g) = B \upharpoonright (\delta+1)$ is defined. For Ω many $\alpha > \beta$, $E(\alpha)$ is $(B \upharpoonright (\gamma+1), \{g\})$. For each such α, by Case 2 of the construction of $F_{\alpha+1}$, there is assigned $h \in (2^\omega)^\gamma$ such that h extends g, $F_{\alpha+1}(h) = B \upharpoonright (\gamma+1)$, and $F_\alpha(h)$ is not defined. Then h is in $T \cap (2^\omega)^\gamma$ and h extends g. Since $F_\alpha(h)$ is not defined, h was not assigned to a smaller α; thus, distinct such α's give distinct h's.

(3) $F(f) \subset B$. Thus $F(f)$ is defined. Since $F \in \mathscr{F}$, $F(f \upharpoonright \beta) \subseteq F(f) \subset B$. Thus, $f \upharpoonright \beta \in T$.

LEMMA 2. *If countable* $t \subseteq T$ *and there are* Ω *many* $h \in (2^\omega)^\gamma$ *such that* h *is a union of elements of* t, *then there exists* $h \in T \cap (2^\omega)^\gamma$ *such that* h *is a union of elements of* t.

PROOF. For each $k \in t$, $F(k) \subset B$ and some $F_\beta(k)$, $\beta < \Omega$, is defined. Since Ω is not cofinal with ω, there exists $\beta < \Omega$ such that, for all $k \in t$, $F_\beta(k)$ is defined. For some $\alpha > \beta$, $E(\alpha)$ is $(B \upharpoonright (\gamma+1), t)$. Then Case 3 of the definition of $F_{\alpha+1}$ gives $h \in (2^\omega)^\gamma$ such that h is a union of elements of t and $F_{\alpha+1}(h) = B \upharpoonright (\gamma+1)$. Then $F(h) \subset B$ and $h \in T$.

Order T by: $f \leqq g$ iff f extends g. Then T satisfies (1) of §1. So there is a (algebraically unique) complete Boolean algebra C such that T is a dense subset of C and the ordering of C extends that of T.

LEMMA 3. *C is an ω-distributive Boolean algebra.*

PROOF. Suppose that A is a function on $\omega \times K$ into C. By the definition of ω-distributivity and by the fact that T is dense in C, it suffices to show for every $f \in T$ that: if $f \leqq \bigwedge_m \bigvee_\alpha A(m, \alpha)$, then there is $h \in T$ such that $h \leqq f$ and, for all $m < \omega$, $h \leqq A(m, \alpha)$ for some $H(m) = \alpha < K$.

Define $P(r) \in T$ for $r \in 2^m$ by the following induction on $m < \omega$. Let $P(0) = f$. Suppose $P(r) \leqq f$ defined for all $r \in 2^m$. Let $\beta(m)$ be the least upper bound of $\{\mathrm{Dm}(P(r)) \mid r \in 2^m\}$. Consider any $r \in 2^m$. Let u and v be the two extensions in 2^{m+1} of r. Since $P(r) \leqq f \leqq \bigvee_\alpha A(m, \alpha)$, there exists $g \in T$ such that $g \leqq P(r)$ and $g \leqq A(m, \alpha)$ for some α. By Lemma 1(2), there exist distinct extensions $P(u)$ and $P(v)$ of g such that they both have the same domain greater than $\beta(m)$.

Put $t = \{P(r) \mid r \in 2^m$ for some $m < \omega\}$. Let $\gamma = \bigcup_m \beta(m)$. By Lemma 2, there exists $h \in T \cap (2^\omega)^\gamma$ such that h is a union of some elements of t. Then $h \leqq P(0) = f$. Consider any $m < \omega$. There is $r \in 2^{m+1}$ such that $P(r) \subset h$. There is $\alpha < K$ such that $h \leqq P(r) \leqq A(m, \alpha)$.

LEMMA 4. *T has an uncountable chain.*

PROOF. Define A on $\omega_1 \times 2$ into C by

$$A(\omega\beta + m, i) = \bigvee \{f \in T \mid f(\beta)(m) = i\}, \quad \beta < \omega_1, m < \omega, i < 2.$$

By Lemma 1(2), for every $g \in T$ there is $f \leqq g$ such that $\beta \in \mathrm{Dm}(f)$ and hence $f \leqq A(\omega\beta + m, f(\beta)(m))$. Thus since T is dense, it follows that each $\bigvee_i A(\omega\beta + m, i)$ is the identity of C. Since C is a complete ω-distributive Boolean algebra, it must have property P_{ω_1} (by our earlier supposition, which is used here only). By P_{ω_1}, there exists a function H on ω_1 into 2 such that, for all $\gamma < \omega_1$, the infimum $\bigwedge_{\beta < \gamma} A(\beta, H(\beta))$ is nonzero. Let $J = \{f \in X \mid f(\beta)(m) = H(\omega\beta + m)$ for all $\beta < \mathrm{Dm}(f)$ and $m < \omega\}$. Then J is an uncountable chain of functions. It now suffices to prove $J \subseteq T$. Consider any $f \in J$. Let $\alpha = \mathrm{Dm}(f)$ and $\gamma = \omega\alpha$. Since T is dense, there exists $g \in T$,

$g \leqq \bigwedge_{\beta<\gamma} A(\beta, H(\beta))$. Then $g(\beta)(m)=H(\omega\beta+m)$ for all $\beta<\alpha$, $m<\omega$; so $g \restriction \alpha$ is f. By Lemma 1(3), $f \in T$.

For each $B \in (2^\omega)^{\omega_1}$, we now choose $T(n, B)$, $J(n, B)$, by induction on $n<\omega$, such that $T(n, B) \subseteq X$, $J(n, B) \in (2^\omega)^{\omega_1}$, and $J(n, B) \restriction \alpha \in T(n, B)$ for all nonzero $\alpha<\omega_1$.

$T(0, B)=X$. $J(0, B)=B$.

Suppose $T(n, B)$ and $J(n, B)$ are defined. Define $T(n+1, B)=\{f \mid F(f) \subset J(n, B)\}$ (i.e., define $T(n+1, B)$ from $J(n, B)$ as T was earlier defined from B). By Lemma 4, $T(n+1, B)$ has an uncountable branch. The union of such a branch is some element $J(n+1, B)$ of $(2^\omega)^{\omega_1}$.

LEMMA 5. *If B and B' are such that, for all $n<\omega$, $J(n, B) \restriction 1$ equals $J(n, B') \restriction 1$, then B equals B'.*

PROOF. Let B and B' be such functions. It suffices to prove, by induction on nonzero $\beta \leqq \omega_1$, that $R(\beta): J(n, B) \restriction \beta = J(n, B') \restriction \beta$ for all $n<\omega$. For $\beta=1$, $R(\beta)$ is given. For nonzero limit β, $R(\beta)$ follows easily from $R(\alpha)$ for all $\alpha<\beta$. Suppose $R(\beta)$ holds for nonzero β, to show $R(\beta+1)$. Consider any $n<\omega$. Let f be $J(n+1, B) \restriction \beta$. By $R(\beta)$, f equals $J(n+1, B') \restriction \beta$. Then $F(f) \subset J(n, B)$ since $f \in T(n+1, B)$; $F(f) \subset J(n, B')$ since $f \in T(n+1, B')$. Since $F(f)$ has domain $\beta+1$, it follows that $J(n, B) \restriction (\beta+1) = F(f) = J(n, B') \restriction (\beta+1)$.

By Lemma 5, distinct $B \in (2^\omega)^{\omega_1}$ are assigned distinct sequences $\langle J(n, B) \restriction 1 \mid n<\omega \rangle$. Thus, $|(2^\omega)^{\omega_1}| \leqq |((2^\omega)^1)^\omega|$. This contradicts the assumption that $|2^{\omega_1}| > |2^\omega|$. Theorem 2 is proved.

It follows from the proof of Theorem 2 that: if $|2^\omega| < |2^{\omega_1}|$, then, for some B, the corresponding T defined above is a dense subset of an ω-distributive complete Boolean algebra not having property P_{ω_1}.

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