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ON RESTRICTED WEAK TYPE (1, 1)¹
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Abstract. Let \( (S_k)_{k \geq 1} \) be a sequence of linear operators defined on \( L^1(\mathbb{R}^n) \) such that for every \( f \in L^1(\mathbb{R}^n) \), \( S_k f = f * g_k \) for some \( g_k \in L^1(\mathbb{R}^n) \), \( k = 1, 2, \ldots \), and \( T f(x) = \sup_{k \geq 1} |S_k f(x)| \). Then the inequality \( m \{ x \in \mathbb{R}^n; T f(x) > y \} \leq C y^{-1} \int_{\mathbb{R}^n} |f(t)| \, dt \) holds for characteristic functions \( f \) (\( T \) is of restricted weak type \((1, 1)\)) if and only if it holds for all functions \( f \in L^1(\mathbb{R}^n) \) (\( T \) is of weak type \((1, 1)\)). In particular, if \( S_k f \) is the \( k \)th partial sum of Fourier series of \( f \), this theorem implies that the maximal operator \( T \) related to \( S_k \) is not of restricted weak type \((1, 1)\).

1. Introduction. We will show that maximal operators of a certain type are of weak type \((1, 1)\) if and only if they are of restricted weak type \((1, 1)\). Many important operators are of the type considered.

Throughout, \( \mathbb{R}^n \) will denote \( n \)-dimensional Euclidean space, \( m \) will denote Lebesgue measure on \( \mathbb{R}^n \), and \( f \) will denote a measurable function on \( \mathbb{R}^n \). Recall that \( L^p(\mathbb{R}^n) \) is the set of all real (or complex) valued measurable functions on \( \mathbb{R}^n \) with the property

\[
\| f \|_p = \left( \int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p} < \infty, \quad 1 \leq p < \infty,
\]

\[
\| f \|_\infty = \inf \{ y; m \{ x \in \mathbb{R}^n; |f(x)| > y \} = 0 \} < \infty.
\]

\( C_c(\mathbb{R}^n) \) will denote the set of all continuous functions on \( \mathbb{R}^n \) with compact supports and \( S(\mathbb{R}^n) \) will denote the set of all simple functions each of which is a finite linear combination of characteristic functions of compact connected sets.

The convolution of measurable functions \( f \) and \( g \) on \( \mathbb{R}^n \) is defined by

\[
(f \ast g)(x) = \int_{\mathbb{R}^n} f(t) g(x - t) \, dt
\]

whenever the integral exists. Note that

\[
\| f \ast g \|_1 \leq \| f \|_1 \cdot \| g \|_1.
\]

Let \( T \) be an operator defined on \( L^p(\mathbb{R}^n) \).

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T is of weak type \((p, q)\) if there exists a positive constant \(A\) such that for each function \(f\) in \(L^p(\mathbb{R}^n)\) and \(y > 0\)

\[(1.4) \quad m\{x \in \mathbb{R}^n; |Tf(x)| > y\} \leq ((A/y) \|f\|_p)^q.\]

\(T\) is of restricted weak type \((p, q)\) if inequality \((1.4)\) holds whenever \(f\) is restricted to the collection of characteristic functions of measurable set in \(\mathbb{R}^n\) with finite measure.

It is obvious that \(T\) is of restricted weak type \((p, q)\) if it is of weak type \((p, q)\). But the converse is not true for \(p > 1\) (see [5]). We will, however, prove that for some special operators the converse is true for \(p = 1\).

2. Restricted weak type \((p, q)\). Stein and Weiss [5] considered the operator \(T\) defined by

\[Tf(x) = x^{-1/p} \int_0^\infty y^{-1/q} f(y) \, dy\]

and showed that \(T\) is of restricted weak type \((p, q)\) but not of weak type \((p, q)\), in the case \(p > 1\), where \(1/p + 1/p' = 1\).

However, we are able to prove the following theorem:

**Theorem.** Let \(S_n\) \((n = 1, 2, \cdots)\) be linear operators on \(L^1(\mathbb{R}^m)\), each of the form \(S_n f = f \ast g_n\) for some \(g_n \in L^1(\mathbb{R}^m)\), and let \(Tf(x) = \sup_{n \geq 1} |S_n f(x)|\).

Then, \(T\) is of restricted weak type \((1, q)\), \(q \geq 1\), if and only if \(T\) is of weak type \((1, q)\).

**Proof.** It is enough to show that \(T\) is of weak type \((1, q)\) if it is of restricted weak type \((1, q)\) since the converse is trivial.

Let \(f \geq 0\) be a function in \(S(\mathbb{R}^m)\) such that \(\|f\|_\infty \neq 0\). Since \(C_c(\mathbb{R}^m)\) is dense in \(L^1(\mathbb{R}^m)\), for any given \(\epsilon > 0\), there exist \(h_n \in C_c(\mathbb{R}^m)\) \((n = 1, 2, \cdots)\) such that

\[(2.1) \quad \|g_n - h_n\|_1 < \epsilon/2 \max(1, \|f\|_\infty).\]

Then we have

\[(2.2) \quad |f \ast g_n(x) - f \ast h_n(x)| \leq \int_{\mathbb{R}^m} |f(t)| |g_n(x - t) - h_n(x - t)| \, dt \leq \|f\|_\infty \|g_n - h_n\|_1 < \frac{\epsilon}{2}.\]

For any fixed \(\lambda > 0\) and all positive integers \(n\), \(1 \leq n \leq N\), there exists \(\delta = \delta(N) > 0\) such that, for any connected set \(I\) with \(\text{dia}(I) = \sup\{|x - y|; x, y \in I\} < \delta\), \(x, y \in I\) implies

\[(2.3) \quad |h_n(x) - h_n(y)| < \lambda/2 \|f\|_1.\]

We now divide \(\mathbb{R}^m\) into disjoint connected sets \(I_k\) such that \(\text{dia}(I_k) < \delta\) and \(f(x) = \alpha_k\) on \(I_k\) where \(\alpha_k\)'s are positive real numbers. Note that such of \(\alpha_k\)'s are finitely many since \(f \in S(\mathbb{R}^m)\). Put \(\alpha = \max\{\alpha_k\}\). Clearly \(\alpha = \|f\|_\infty\).
Let $F_k$ be a subinterval of $I_k$ such that $m(F_k) = (\alpha_k/x)m(I_k)$ and set $E_N = \bigcup_k F_k$.

Thus, we have
\begin{equation}
\alpha m(E_N) = \sum_k \alpha m(F_k) = \sum_k \alpha_k m(I_k) = \|f\|_1.
\end{equation}

Combining with (2.3) and applying the mean values theorem, we obtain, for each $n$, $1 \leq n \leq N$,
\begin{align*}
|f \ast h_n(x) - \alpha \chi_{E^*} \ast h_n(x)| &= \left| \int_{E^*} f(t) h_n(x - t) \, dt - \alpha \int_{E^*} h_n(x - t) \, dt \right| \\
&\leq \sum_k \left| \int_{I_k} h_n(x - t) \, dt - \alpha \int_{F_k} h_n(x - t) \, dt \right| \\
&= \sum_k |\alpha_k m(I_k) h_n(x - t_k) - \alpha m(F_k) h_n(x - t'_k)| \\
&= \sum_k m(F_k) \frac{\lambda}{2 \|f\|_1} = \frac{\lambda}{2}.
\end{align*}

A combination of (2.2), (2.5), and (2.1) with $\alpha = \|f\|_\infty$ gives, for each $n$, $1 \leq n \leq N$,
\begin{align*}
|S_n f(x) - \alpha S_n \chi_{E^*} f(x)| &\leq |f \ast g_n(x) - f \ast h_n(x)| \\
&+ |f \ast h_n(x) - \alpha \chi_{E^*} \ast h_n(x)| \\
&+ \alpha |\chi_{E^*} \ast h_n(x) - \chi_{E^*} \ast g_n(x)| \\
&\leq \lambda/2 + \epsilon.
\end{align*}

Hence, we obtain
\begin{equation}
T_N f(x) = \sup_{1 \leq n \leq N} |S_n f(x)| \leq \alpha T_N \chi_{E^*} f(x) + \lambda/2 + \epsilon.
\end{equation}

From (2.4) and the fact that $T$ is of restricted weak type $(1, q)$, (2.6) implies
\begin{align*}
m\{x \in \mathbb{R}^m; T_N f(x) > \lambda + \epsilon\} &\leq m\{x \in \mathbb{R}^m; T \chi_{E^*} f(x) > \lambda + 2\alpha\} \\
&\leq ((A/\lambda) \alpha m(E_N))^q = ((A/\lambda) \|f\|_1)^q.
\end{align*}

Since $T_N f(x) \leq T_{N+1} f(x)$ for all $x \in \mathbb{R}^m$ and $\epsilon > 0$ is arbitrary, we finally get
\begin{align*}
m\{x \in \mathbb{R}^m; T f(x) > \lambda\} &= \lim_{N \to \infty} m\{x \in \mathbb{R}^m; T_N f(x) > \lambda\} \\
&\leq ((A/\lambda) \|f\|_1)^q \quad \text{for all } f \in S(\mathbb{R}^m).
\end{align*}

We now consider a general function $f$ in $L^1(\mathbb{R}^m)$. Let $N$ be a fixed positive integer. For any given $\epsilon > 0$, there exists a function $h_N \in S(\mathbb{R}^m)$
such that
\[ \|f - h_N\|_1 < \varepsilon^2/\max(1, M) \]
where \( M = \max_{1 \leq n \leq N} \|g_n\|_1 \). Then, for each \( n \), \( 1 \leq n \leq N \), we have
\[ \|S_n f - S_n h_N\|_1 \leq \|g_n\|_1 \|f - h_N\|_1 < \varepsilon^2 \]
and
\[ m\{x \in \mathbb{R}^m ; |S_n f(x) - S_n h_N(x)| > \varepsilon \} < \varepsilon. \]

Denote \( B_n(N) = \{x \in \mathbb{R}^m ; |S_n f(x) - S_n h_N(x)| > \varepsilon \} \) and \( B_N = \bigcup_{n=1}^N B_n(N) \). Then, for all \( x \notin B_N \) and \( n=1, 2, \cdots, N \),
\[ T_N f(x) = \sup_{1 \leq n \leq N} |S_n f(x)| \leq T_N h_N(x) + \varepsilon \leq T h_N(x) + \varepsilon. \]

From (2.7), (2.8), and (2.9), we get
\[ m\{x \in \mathbb{R}^m ; T_N f(x) > \lambda + \varepsilon \} \leq m\{x \in \mathbb{R}^m ; Th_N(x) > \lambda \} + mB_N \]
\[ \leq \left( \frac{A}{\lambda} \|f\|_1 \right)^q + \sum_{n=1}^N mB_n(N) \]
\[ \leq \left( \frac{A}{\lambda} \|f\|_1 + \varepsilon \right)^q + N\varepsilon. \]

Since \( \varepsilon \) is arbitrary, we obtain
\[ m\{x \in \mathbb{R}^m ; T_N f(x) > \lambda \} \leq \left( \frac{A}{\lambda} \|f\|_1 \right)^q \]
and finally
\[ m\{x \in \mathbb{R}^m ; T f(x) > \lambda \} = \lim_{N \to \infty} m\{x \in \mathbb{R}^m ; T_n f(x) > \lambda \} \leq \left( \frac{A}{\lambda} \|f\|_1 \right)^q. \]

This completes the theorem.

3. Applications. Let \( S_n f(x) \) be the \( n \)th partial sum of the Fourier series of \( f(x) \) with respect to a complete orthonormal system \( \{\phi_n; n=0, 1, 2, \cdots\} \) defined on a measurable set \( G \) in \( \mathbb{R} \), that is,
\[ S_n f(x) = \sum_{j=0}^{n-1} \phi_j(x) \int_G f(t) \phi_j(t) dt \]
and let
\[ M f(x) = \sup_{n \geq 1} |S_n f(x)|. \]

We will denote by \( \Phi(L) \) the set of all measurable functions \( f \) on \( G \) such that
\[ \int_G \Phi(|f(x)|) \, dx < \infty \]
and \( \log^+ x = \max(0, \log x) \).

On the trigonometric system and the Walsh-Paley system, Sjölin [4] has shown that for each function \( f \) in the class \( L(\log^+ L)(\log^+ \log^+ L) \),
$S_nf(x)$ converges almost everywhere (a.e.) to $f(x)$ by using the fact that $M$ is of restricted weak type $(p, p), 1 < p < \infty$ (so called “the basic result”) ([2] and [4]). We also know that there exists a function $f$ in the class $L(\log^+ \log^+ L)^{1-\epsilon}$ for $\epsilon > 0$ such that $S_nf(x)$ diverges a.e. ([1] and [3] for the trigonometric system and [3] for the Walsh-Paley system).

The convergences or divergences of the functions in the classes between $L(\log^+ L)(\log^+ \log^+ L)$ and $L(\log^+ \log^+ L)$ for both systems are open questions.

Suppose that $M$ were of restricted weak type $(1, 1)$. Then, by following the same proof of the a.e. convergence of functions in $L(\log^+ L)(\log^+ \log^+ L)$ [4], we would be able to prove that for each function $f$ in the class $L(\log^+ \log^+ L)$, $S_nf(x)$ converges a.e. to $f(x)$. But unfortunately we know that for both systems, $M$ is not of weak type $(1, 1)$ and so is not of restricted weak type $(1, 1)$ by our theorem. This shows that the modification of the method in [2] and [4] to prove the almost everywhere convergence of functions in the class $L(\log^+ \log^+ L)$ is not available.

Let us note that the maximal Hilbert transform $M$ defined by

\begin{equation}
Mf(x) = \sup_{n \geq 1} |H_nf(x)|,
\end{equation}

where $H_nf(x) = \int_{1/n < |x-t| < n} f(t)/(x-t) dt$, is of the type that we have considered.

The Hardy-Littlewood maximal operator $\Lambda$ defined by

\begin{equation}
\Lambda f(x) = \sup_{n \geq 1} \left( \frac{1}{|I_n(x)|} \int_{I_n(x)} |f(t)| dt \right),
\end{equation}

where $I_n(x)$ is any interval with center at $x$ and length $2^{-n}$ is essentially of this type.

**REFERENCES**