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ON TWO THEOREMS OF PALEY[1]

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ABSTRACT. A strengthening of Paley's theorem for the Fourier coefficients of an L^p function is presented. The result is then applied to prove strong versions of recent results of P. L. Duren, and of J. H. Hedlund on (L^p, L^q) multipliers.

Introduction. Recently J. H. Hedlund [2] has proved the following theorem: *If* $\{\lambda(n)\}$ *satisfies*: $\sup_{0\leqq k}(\sum_{n\in B_k}|\lambda(n)|^q)^{1/q}<\infty$, *where*

$$\begin{aligned} B_k &= \{0\}, \qquad k=0, \\ &= \{n\in \boldsymbol{Z} \mid 2^{k-1}\leqq n<2^k\}, \end{aligned}$$

then $\{\lambda(n)\}$ *is an* (H^p, H^q) *multiplier, where* $1\leqq p\leqq 2$, $q=2p/(2-p)$.

This result implies a sufficient condition given by Hardy and Littlewood (see [2, Proposition 5]). The result for $p=2$ is of course trivial, and in the case $p=1$ is due to Hardy and Littlewood. Actually the condition for $p=1$ is necessary as well as sufficient, as was proved by Stein and Zygmund in [7].

Using Hedlund's result, Kellog [4] has proved the following improvement of the Hausdorff-Young theorem: *If* $1<p\leqq 2, f\in L^p$, *then*:

$$\left(\sum_{k=-\infty}^{\infty}\left(\sum_{n\in B_k}|\hat{f}(n)|^{p'}\right)^{2/p'}\right)^{1/2}\leqq C_p\,\|f\|_{L^p}$$

where $B_k=-B_{-k}$ *for* $k<0$ *and* $1/p+1/p'=1$. *If*

$$\left(\sum_{k=-\infty}^{\infty}\left(\sum_{n\in B_k}|C_n|^p\right)^{2/p}\right)^{1/2}<\infty,$$

then $f\in L^{p'}$ *exists so that*

$$C_n=\hat{f}(n) \quad \textit{and} \quad \|f\|_{L^{p'}}\leqq C_p\left(\sum_{k=-\infty}^{\infty}\left(\sum_{n\in B_k}|C_n|^p\right)^{2/p}\right)^{1/2}.$$

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Since however the Hausdorff-Young theorem is not the best one can say about the Fourier coefficients of a function, it is of interest to compare Kellog's theorem with Paley's: $(\sum_1^\infty [\hat{f}(n)^*]^p n^{p-2})^{1/p} \leqq C_p \|f\|_{L^p}$ *where* $\{f(n)^*\}$ *is the nonincreasing rearrangement of* $\{|f(n)|\}$. If one considers Fourier coefficients with respect to a general uniformly bounded orthonormal system, Paley's theorem is a best possible one (see [8]). However, in the case of the trigonometric system, we see that Kellog's result is not comparable to Paley's: If $\{f(n)\}$ are lacunary, Kellog's result is better, while if $\{\hat{f}(n)\}$ vanishes except on a binary block B_k, Paley's theorem is better.

In this note we prove a theorem which is an improvement of both Kellog's and Paley's theorems. The proof is surprisingly simple. Using this result we in turn improve Hedlund's multiplier theorem, as well as a multiplier theorem of Duren.

To keep the presentation simple we restrict ourselves to periodic functions. The extensions to R^n are straightforward.

$L(p, q)$ spaces are defined as follows: (X, Σ, μ) is a σ-finite measure space, f a measurable function, f^* the nonincreasing rearrangement of f. Define

$$\|f\|^*_{p,q} = \left(\int_0^\infty [t^{1/p} f^*(t)]^q \frac{dt}{t}\right)^{1/q}, \qquad 0 < p < \infty, \qquad 0 < q < \infty,$$

$$= \sup_{0<t} t^{1/p} f^*(t), \qquad 0 < p \leqq \infty, \qquad q = \infty.$$

$$L(p, q) = \{f \mid \|f\|^*_{p,q} < \infty\}.$$

For a survey of the theory of $L(p, q)$ spaces and their interpolation properties, see for example [3], [6]. We mention only the facts most important in the present context.

(a) $q_1 < q_2 \to L(p, q) \subset L(p, q_2)$, and the inclusion is strict, unless (X, Σ, μ) has only finitely many disjoint sets of positive measure.

(b) $L(p, p) = L^p$.

(c) For sequences $\{a_n\}$, considered as functions over the integers, with measure 1 on each integer,

$$\|\{a_n\}\|^*_{p,q} \sim \left(\sum_1^\infty (a_n^*)^q n^{q/p-1}\right)^{1/q}$$

where $\{a_n^*\}$ is the nonincreasing rearrangement of $\{|a_n|\}$.

(d) (*Hölder's inequality*)

$$\|fg\|^*_{p,q} \leqq e^{1/p} \|f\|^*_{p_0,q_0} \|g\|^*_{p_1,q_1}$$

where $1/p = 1/p_0 + 1/p_1$; $1/q = 1/q_0 + 1/q_1$.

We next quote two theorems:

THEOREM I (PALEY). *If $\{\varphi_n\}$ is a uniformly bounded orthonormal system, $f(n)$ the Fourier coefficients of f with respect to this system, $1<p<2$, $0<q\leqq\infty$,*

$$\|\{\hat{f}(n)\}\|^*_{p',q} \leqq C_{p,q}\,\|f\|^*_{p,q}.$$

If $\{C_n\}\in L(p,q)$, then $f\in L(p',q)$ exists so that $C_n=\hat{f}(n)$, and

$$\|f\|^*_{p',q}\leqq C_{p,q}\,\|\{C_n\}\|^*_{p,q}.$$

See [3], [6]. The case $q=p$ is the classical theorem of Paley mentioned in the introduction.

THEOREM II (LITTLEWOOD-PALEY). $f\sim\sum_{-\infty}^{\infty}\hat{f}(n)e^{inx}$, $f\in L^p$, $1<p<\infty$. Denote $\Delta_k(x)=\sum_{n\in B_k}\hat{f}(n)e^{inx}$. *Then*

$$C_p\,\|f\|_{L^p}\leqq\left\|\left(\sum_{-\infty}^{\infty}|\Delta_k(x)|^2\right)^{1/2}\right\|_{L^p}\leqq C_p\,\|f\|_{L^p}.$$

See [5], [8].

THEOREM III. *Let $f\sim\sum_{-\infty}^{\infty}\hat{f}(n)e^{inx}$, $f\in L^p$, $1<p\leqq 2$. Then, using the above notation,*

$$\left(\sum_{k=-\infty}^{\infty}\|\hat{\Delta}_k(n)\|^{*2}_{p',p}\right)^{1/2}\leqq C_p\,\|f\|_{L^p}.$$

Conversely, if $2\leqq p<\infty$ and $(\sum_{k=-\infty}^{\infty}\|\hat{\Delta}_k(n)\|^{*2}_{p',p})^{1/2}<\infty$, there exists $f\in L^p$ such that $f\sim\sum_{k=-\infty}^{\infty}(\sum_{B_k}\hat{\Delta}_k(n)e^{inx})$.

PROOF. If $1<p\leqq 2$,

$$\begin{aligned}C_p\,\|f\|_{L^p}&\geqq\left\|\left(\sum_{-\infty}^{\infty}|\Delta_k(x)|^2\right)^{1/2}\right\|_{L^p}\\&\geqq\left(\sum_{k=-\infty}^{\infty}\|\Delta_k(x)\|^2_{L^p}\right)^{1/2}\geqq\left(\sum_{k=-\infty}^{\infty}\|\hat{\Delta}_k(n)\|^{*2}_{p',p}\right)^{1/2}.\end{aligned}$$

Of course $\hat{\Delta}_k(n)=\hat{f}(n)$ if $n\in B_k$, $\hat{\Delta}_k(n)=0$ otherwise.

For the other part, note

$$\begin{aligned}C_p\,\|f\|_{L^p}&\leqq\left\|\left(\sum_{-\infty}^{\infty}|\Delta_k(x)|^2\right)^{1/2}\right\|_{L^p}\\&\leqq\left(\sum_{-\infty}^{\infty}\|\Delta_k(x)\|^2_{L^p}\right)^{1/2}\leqq\left(\sum_{-\infty}^{\infty}\|\hat{\Delta}_k(n)\|^{*2}_{p',p}\right)^{1/2}.\end{aligned}$$

(To be precise, given a sequence $\{C_n\}$ with $(\sum_{-\infty}^{\infty}\|\{C_n\}_{n\in B_k}\|^{*2}_{p',p})^{1/2}<\infty$, we define $f_N(x)=\sum_{|n|\leqq N}C_ne^{inx}$, apply the norm inequality above, and use completeness of L^p.)

A moment's reflection shows that the result above is an improvement of both Kellog's theorem and of Paley's. If one uses Hausdorff-Young in conjunction with Theorem III, one gets precisely Kellog's theorem. To show the improvements of the multiplier theorems of Hedlund and Kellog, we introduce the following notation

$$\|\{C_n\}\|_{l^r(l^{p,q})} = \left(\sum_{-\infty}^{\infty} \|\{C_n\}_{n\in B_k}\|_{p,q}^{*r}\right)^{1/r},$$

$$l^r(l^{p,q}) = \{\{C_n\} \mid \|\{C_n\}\|_{l^r(l^{p,q})} < \infty\}.$$

If now $1<p\leqq 2\leqq q<\infty$, $1/r=1/p-1/q$, $\Lambda=\{\lambda(n)\}\in l^\infty(l^{r,\infty})$, we have

$$\begin{aligned}\|\{\lambda(n)\hat{f}(n)\}_{n\in B_k}\|_{q',q}^{*} &\leqq \|\{\lambda(n)\hat{f}(n)\}_{n\in B_k}\|_{q',p}^{*}\\ &\leqq e^{1/q'}\,\|\{\lambda(n)\}_{n\in B_k}\|_{r,\infty}\,\|\{\hat{f}(n)\}_{n\in B_k}\|_{p',p}.\end{aligned}$$

Hence

THEOREM IV. *If $\{\lambda(n)\}\in l^\infty(l^{r,\infty})$, $1/r=1/p-1/q$, $1<p\leqq 2\leqq q<\infty$, then $\Lambda(f)$ defined as $(\Lambda f)\hat{\ }(n)=\lambda(n)\hat{f}(n)$ is a bounded mapping from L^p into L^q*

PROOF.

$$\begin{aligned}\|\Lambda f\|_{L^q} &\leqq C_q\,\|\{\lambda(n)\hat{f}(n)\}\|_{l^2(l^{q,q'})}\\ &\leqq C_{p,q}\,\|\{\lambda(n)\}\|_{l^\infty(l^{r,\infty})}\,\|\{\hat{f}(n)\}\|_{l^2(l^{p',p})}\\ &\leqq C_{p,q}\,\|\{\lambda(n)\}\|_{l^\infty(l^{r,\infty})}\,\|f\|_{L^p}.\end{aligned}$$

The following theorem of Duren [1] is an easy consequence of Theorem IV.

If $\lambda_n=O(n^{-\alpha})$ where $\alpha=1/p-1/q$, $1<p\leqq 2\leqq q$. Then $\{\lambda_n\}$ is an (L^p, L^q) multiplier. It suffices to observe that if $\lambda_n=O(n^{-\alpha})$, we have $\lambda_n^*\leqq C/n^\alpha$, then $\{\lambda_n\}\in l(1/\alpha,\infty)$, and certainly $\{\lambda_n\}\in l^\infty(l^{1/\alpha,\infty})$ as required by our multiplier theorem. Clearly we can prove a stronger version of Duren's theorem: *If $|\lambda_n|\leqq C(n-2^k)^{-\alpha}$ where $2^k<n\leqq 2^{k+1}$ and C is uniform in k, then $\{\lambda_n\}$ is an (L^p, L^q) multiplier.*

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