ON TWO THEOREMS OF PALEY

N. M. RIVIERE AND Y. SAGHER

Abstract. A strengthening of Paley's theorem for the Fourier coefficients of an $L^p$ function is presented. The result is then applied to prove strong versions of recent results of P. L. Duren, and of J. H. Hedlund on $(L^p, L^q)$ multipliers.

Introduction. Recently J. H. Hedlund [2] has proved the following theorem: If $\{\lambda(n)\}$ satisfies: $\sup_{0 \leq k} (\sum_{n \in B_k} |\lambda(n)|^q)^{1/q} < \infty$, where

$$B_k = \begin{cases} \{0\}, & k = 0, \\ \{ n \in \mathbb{Z} | 2^{k-1} \leq n < 2^k \}, & \end{cases}$$

then $\{\lambda(n)\}$ is an $(H^p, H^q)$ multiplier, where $1 \leq p \leq 2$, $q = 2p/(2-p)$.

This result implies a sufficient condition given by Hardy and Littlewood (see [2, Proposition 5]). The result for $p=2$ is of course trivial, and in the case $p=1$ is due to Hardy and Littlewood. Actually the condition for $p=1$ is necessary as well as sufficient, as was proved by Stein and Zygmund in [7].

Using Hedlund’s result, Kellog [4] has proved the following improvement of the Hausdorff-Young theorem: If $1 < p \leq 2$, $f \in L^p$, then:

$$\left( \sum_{k=-\infty}^{\infty} \left( \sum_{n \in B_k} |f(n)|^p \right)^{2/p} \right)^{1/2} \leq C_p \| f \|_{L^p}$$

where $B_k = -B_{-k}$ for $k < 0$ and $1/p + 1/p' = 1$. If

$$\left( \sum_{k=-\infty}^{\infty} \left( \sum_{n \in B_k} |C_n|^p \right)^{2/p} \right)^{1/2} < \infty,$$

then $f \in L^{p'}$ exists so that

$$C_n = f(n) \quad \text{and} \quad \| f \|_{L^{p'}} \leq C_p \left( \sum_{k=-\infty}^{\infty} \left( \sum_{n \in B_k} |C_n|^p \right)^{2/p} \right)^{1/2}.$$
Since however the Hausdorff-Young theorem is not the best one can say about the Fourier coefficients of a function, it is of interest to compare Kellog's theorem with Paley's: \((\sum_in \{f(n)\}^p n^{p-2} 1/p \leq C_p \|f\|_L^p\) where \(\{f(n)\}\) is the nonincreasing rearrangement of \(|f(n)|\). If one considers Fourier coefficients with respect to a general uniformly bounded orthonormal system, Paley's theorem is a best possible one (see [8]). However, in the case of the trigonometric system, we see that Kellog's result is not comparable to Paley's: If \(\{f(n)\}\) are lacunary, Kellog's result is better, while if \(\{f(n)\}\) vanishes except on a binary block \(B_k\), Paley's theorem is better.

In this note we prove a theorem which is an improvement of both Kellog's and Paley's theorems. The proof is surprisingly simple. Using this result we in turn improve Hedlund's multiplier theorem, as well as a multiplier theorem of Duren.

To keep the presentation simple we restrict ourselves to periodic functions. The extensions to \(R^n\) are straightforward.

\(L(p, q)\) spaces are defined as follows: \((X, \Sigma, \mu)\) is a \(\sigma\)-finite measure space, \(f\) a measurable function, \(f^*\) the nonincreasing rearrangement of \(f\). Define

\[
\|f\|_{p, q}^* = \left(\int_0^\infty \left(t^{1/p} f^*(t)\right)^q dt\right)^{1/q}, \quad 0 < p < \infty, \quad 0 < q < \infty,
\]

\[
= \sup_{t>0} t^{1/p} f^*(t), \quad 0 < p \leq \infty, \quad q = \infty.
\]

\(L(p, q) = \{f | \|f\|_{p, q}^* < \infty\}\).

For a survey of the theory of \(L(p, q)\) spaces and their interpolation properties, see for example [3], [6]. We mention only the facts most important in the present context.

(a) \(q_1 < q_2 \Rightarrow L(p, q_1) \subset L(p, q_2)\), and the inclusion is strict, unless \((X, \Sigma, \mu)\) has only finitely many disjoint sets of positive measure.

(b) \(L(p, p) = L^p\).

(c) For sequences \(\{a_n\}\), considered as functions over the integers, with measure 1 on each integer,

\[
\|\{a_n\}\|_{p, q}^* \sim \left(\sum_1^\infty (a_n^*)^p n^{p-1}\right)^{1/q}
\]

where \(\{a_n^*\}\) is the nonincreasing rearrangement of \(|a_n|\).

(d) (Hölder's inequality)

\[
\|fg\|_{p, q}^* \leq \|f\|_{p_0, q_0}^* \|g\|_{p_1, q_1}^*,
\]

where \(1/p = 1/p_0 + 1/p_1; 1/q = 1/q_0 + 1/q_1\).
We next quote two theorems:

**Theorem I (Paley).** If \( \{q_n\} \) is a uniformly bounded orthonormal system, \( f(n) \) the Fourier coefficients of \( f \) with respect to this system, \( 1 < p < 2, 0 < q \leq \infty \),

\[
\| \{f(n)\} \|_{p,q}^* \leq C_{p,q} \| f \|_{p,q}^*.
\]

If \( \{C_n\} \in L(p,q) \), then \( f \in L(p',q) \) exists so that \( C_n = f(n) \), and

\[
\| f \|_{p',q}^* \leq C_{p,q} \| \{C_n\} \|_{p,q}^*.
\]

See [3], [6]. The case \( q = p \) is the classical theorem of Paley mentioned in the introduction.

**Theorem II (Littlewood-Paley).** \( f \sim \sum_{-\infty}^{\infty} f(n) e^{inz}, f \in L^p, 1 < p < \infty \). Denote \( \Delta_k(x) = \sum_{n \in B_k} f(n) e^{inz} \). Then

\[
C_p \| f \|_{L^p} \leq \left\| \left( \sum_{-\infty}^{\infty} |\Delta_k(x)|^2 \right)^{1/2} \right\|_{L^p} \leq C_p \| f \|_{L^p}.
\]

See [5], [8].

**Theorem III.** Let \( f \sim \sum_{-\infty}^{\infty} f(n) e^{inz}, f \in L^p, 1 < p \leq 2 \). Then, using the above notation,

\[
\left( \sum_{k=-\infty}^{\infty} \| \hat{\Delta}_k(n) \|_{p',q}^* \right)^{1/2} \leq C_p \| f \|_{L^p}.
\]

Conversely, if \( 2 \leq p < \infty \) and \( (\sum_{k=-\infty}^{\infty} \| \hat{\Delta}_k(n) \|_{p',q}^*)^{1/2} < \infty \), there exists \( f \in L^p \) such that \( f \sim \sum_{k=-\infty}^{\infty} (\sum_{n \in B_k} \hat{\Delta}_k(n) e^{inz}) \).

**Proof.** If \( 1 < p \leq 2 \),

\[
C_p \| f \|_{L^p} \geq \left\| \left( \sum_{-\infty}^{\infty} |\Delta_k(x)|^2 \right)^{1/2} \right\|_{L^p} \geq \left( \sum_{k=-\infty}^{\infty} \| \Delta_k(x) \|_{L^p}^2 \right)^{1/2} \geq \left( \sum_{k=-\infty}^{\infty} \| \hat{\Delta}_k(n) \|_{p',q}^* \right)^{1/2}.
\]

Of course \( \hat{\Delta}_k(n) = f(n) \) if \( n \in B_k \), \( \hat{\Delta}_k(n) = 0 \) otherwise.

For the other part, note

\[
C_p \| f \|_{L^p} \leq \left\| \left( \sum_{-\infty}^{\infty} |\Delta_k(x)|^2 \right)^{1/2} \right\|_{L^p} \leq \left( \sum_{-\infty}^{\infty} \| \Delta_k(x) \|_{L^p}^2 \right)^{1/2} \leq \left( \sum_{k=-\infty}^{\infty} \| \hat{\Delta}_k(n) \|_{p',q}^* \right)^{1/2}.
\]

(To be precise, given a sequence \( \{C_n\} \) with \( (\sum_{k=-\infty}^{\infty} \| \{C_n\} \|_{p',q}^*)^{1/2} < \infty \), we define \( f_N(x) = \sum_{|n| \leq N} C_n e^{inz} \), apply the norm inequality above, and use completeness of \( L^p \).)
A moment’s reflection shows that the result above is an improvement of both Kellog’s theorem and of Paley’s. If one uses Hausdorff-Young in conjunction with Theorem III, one gets precisely Kellog’s theorem.

To show the improvements of the multiplier theorems of Hedlund and Kellog, we introduce the following notation

$$\|\{C_n\}\|_{l^p(\ell^{r,s})} = \left(\sum_{n=1}^{\infty} \|\{C_n\}\|_{l^p(\ell^{r,s})}^2\right)^{1/2}$$

$$l^p(\ell^{r,s}) = \{\{C_n\} \mid \|\{C_n\}\|_{l^p(\ell^{r,s})} < \infty\}.$$ 

If now $1 < p \leq 2 \leq q < \infty$, $1/r = 1/p - 1/q$, $\Lambda = \{\lambda(n)\} \in l^\infty(l^{1/\infty})$, we have

$$\|\{\lambda(n)f(n)\}_{n,\ell^{p,s}}\|_{l^q} \leq \|\{\lambda(n)f(n)\}_{l^p(\ell^{r,s})}\|_{l^q} \leq e^{1/q'} \|\{\lambda(n)f(n)\}_{l^p(\ell^{r,s})}\|_{l^q}.$$ 

Hence

**Theorem IV.** If $\{\lambda(n)\} \in l^\infty(l^{1/\infty})$, $1/r = 1/p - 1/q$, $1 < p \leq 2 \leq q < \infty$, then $\Lambda(f)$ defined as $(\Lambda f)(n) = \lambda(n)f(n)$ is a bounded mapping from $L^p$ into $L^q$.

**Proof.**

$$\|\Lambda f\|_{L^q} \leq C_q \|\{\lambda(n)f(n)\}\|_{l^p(\ell^{r,s})} \leq C_{p,q} \|\{\lambda(n)\}\|_{l^{1/\infty}(\ell^{1/\infty})} \|\{f(n)\}\|_{l^p(\ell^{r,s})} \leq C_{p,q} \|\{\lambda(n)\}\|_{l^{1/\infty}(\ell^{1/\infty})} \|\{f(n)\}\|_{l^p(\ell^{r,s})}.$$ 

The following theorem of Duren [1] is an easy consequence of Theorem IV.

**Theorem IV.** If $\lambda_n = O(n^{-\alpha})$ where $\alpha = 1/p - 1/q$, $1 < p \leq 2 \leq q$. Then $\{\lambda_n\}$ is an $(L^p, L^q)$ multiplier. It suffices to observe that if $\lambda_n = O(n^{-\alpha})$, we have $\lambda_n \leq C/n^\alpha$, then $\lambda_n \in l^1(1/\alpha, \infty)$, and certainly $\{\lambda_n\} \in l^\infty(l^{1/\alpha, \infty})$ as required by our multiplier theorem. Clearly we can prove a stronger version of Duren’s theorem: if $|\lambda_n| \leq C(n - 2^k)^{-\alpha}$ where $2^k < n \leq 2^{k+1}$ and $C$ is uniform in $k$, then $\{\lambda_n\}$ is an $(L^p, L^q)$ multiplier.

**References**


**DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455**

**DEPARTMENT OF MATHEMATICS, WEIZMANN INSTITUTE OF SCIENCE, REHOVOT, ISRAEL**