MUTUAL EXISTENCE OF PRODUCT INTEGRALS

JON C. HELTON

Abstract. Definitions and integrals are of the subdivision-refinement type, and functions are from \( R \times R \) to \( R \), where \( R \) represents the real numbers. Let \( OM^0 \) be the class of functions \( G \) such that \( \int_a^b (1+G) \) exists for \( a \leq x < y \leq b \) and \( \int_a^b |1+G-\prod_{i=1}^n (1+G_i)| = 0 \). Let \( OP^0 \) be the class of functions \( G \) such that \( |\prod_{i=1}^n (1+G_i)| \) is bounded for refinements \( \{x_i\}_{i=0}^n \) of a suitable subdivision of \( [a, b] \). If \( F \) and \( G \) are functions from \( R \times R \) to \( R \) such that \( F \in OP^0 \) on \( [a, b] \), \( \lim_{x \to p^+} F(x, y) \) and \( \lim_{y \to p^-} F(x, y) \) exist and are zero for \( p \in [a, b] \), each of \( \lim_{x \to p^+} G(p, x) \) and \( \lim_{x \to p^-} G(p, x) \) exist for \( p \in [a, b] \), and \( G \) has bounded variation on \( [a, b] \), then any two of the following statements imply the other: (1) \( F+G \in OM^0 \) on \( [a, b] \), (2) \( F \in OM^0 \) on \( [a, b] \), and (3) \( G \in OM^0 \) on \( [a, b] \).

All integrals and definitions are of the subdivision-refinement type, and functions are from \( R \times R \) to \( R \), where \( R \) represents the set of real numbers. Furthermore, functions are assumed to be defined only for elements \( \{x, y\} \) of \( R \times R \) such that \( x < y \). If \( D = \{x_i\}_{i=0}^n \) is a subdivision of \( [a, b] \), then \( D(I) = \{[x_{j-1}, x_j]\}_{j=1}^n \) and \( G_{\epsilon} = G(x_{\epsilon-1}, x_{\epsilon}) \). Further, \( \{x_{\epsilon r}\}_{r=0}^n \) represents a subdivision of the interval \( [x_{\epsilon-1}, x_{\epsilon}] \) and \( G_{\epsilon r} = G(x_{\epsilon r-1}, x_{\epsilon r}) \). The statement that \( \int_a^b G \) exists means there exists a number \( L \) such that, if \( \epsilon > 0 \), then there exists a subdivision \( D \) of \( [a, b] \) such that if \( J \) is a refinement of \( D \), then

\[
|L - \sum_{J(I)} G| < \epsilon.
\]

The statement that \( \prod_{J(I)} (1+G) \) exists means there exists a number \( L \) such that, if \( \epsilon > 0 \), then there exists a subdivision \( D \) of \( [a, b] \) such that if \( J \) is a refinement of \( D \), then

\[
|L - \prod_{J(I)} (1+G)| < \epsilon.
\]

Further, \( G \in OA^0 \) on \( [a, b] \) only if \( \int_a^b G \) exists and \( \int_a^b |G - \int G| = 0 \), and

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$G \in OM^\circ$ on $[a, b]$ only if $\prod_{x < y \leq b} (1 + G)$ exists for $a \leq x < y \leq b$ and $\int_a^b |1 + G - \prod_{x < y \leq b} (1 + G)| = 0$.

The statements that $G$ is bounded, $G \in OP^\circ$, $G \in OQ^\circ$ and $G \in OB^\circ$ on $[a, b]$ mean there exist a subdivision $D$ of $[a, b]$ and positive numbers $B$ and $\beta$ such that if $J = \{x_n\}_{n=0}^n$ is a refinement of $D$, then

1. $|G(u)| < B$ for $u \in J(I),$
2. $|\prod_{r=s}^n (1 + G_r)| < B$ for $1 \leq r \leq s \leq n,$
3. $|\prod_{r=s}^n (1 + G_r)| > \beta$ for $1 \leq r \leq s \leq n,$ and
4. $\sum_{I \in J} |G| < B,$ respectively.

If $G$ is a function, then $G \in S_1$ on $[a, b]$ only if $\lim_{x, y \to p^+} G(x, y)$ and $\lim_{z \to p^+} G(x, y)$ exist and are zero for $p \in [a, b]$, and $G \in S_2$ on $[a, b]$ only if $\lim_{z \to p^-} G(p, x)$ and $\lim_{z \to p^-} G(x, p)$ exist for $p \in [a, b]$. Further, $G \in OL^\circ$ on $[a, b]$ only if $\lim_{x, y \to p^+} G(x, y)$, $\lim_{x, y \to p^-} G(x, y)$, $\lim_{x \to p^-} G(p, x)$ and $\lim_{x \to p^-} G(x, p)$ exist for $p \in [a, b]$. See B. W. Helton [2] and J. S. MacNerney [7] for additional details.

**Lemma 1.1.** If $F$ and $G$ are functions from $R \times R$ to $R$ such that $F \in OP^\circ$ on $[a, b]$ and $G \in OB^\circ$ on $[a, b]$, then $F + G \in OP^\circ$ on $[a, b]$.

Lemma 1.1 is part of a previous result by the author [5, Theorem 1].

**Lemma 1.2.** If $G$ is a function from $R \times R$ to $R$ such that $\int_a^b G$ exists, then $G \in OA^\circ$ on $[a, b]$.

Lemma 1.2 is due to A. Kolmogoroff [6, p. 669]. The reader is also referred to results by W. D. L. Appling [1, Theorems 1, 2, p. 155] and B. W. Helton [2, Theorem 4.1, p. 304].

**Lemma 1.3.** If $G$ is a function from $R \times R$ to $R$ such that $G \in OB^\circ$ on $[a, b]$, then the following statements are equivalent:

1. $G \in OM^\circ$ on $[a, b],$
2. $G \in OA^\circ$ on $[a, b], and$
3. $\int_a^b G$ exists.

B. W. Helton [2, Theorem 3.4, p. 301] shows that (1) and (2) are equivalent. Further, by Lemma 1.2, (2) and (3) are equivalent.

**Lemma 1.4.** If $F$ and $G$ are functions from $R \times R$ to $R$ such that $F \in OM^\circ$, $OP^\circ$ and $S_1 \cap S_2$ on $[a, b]$ and $G \in OM^\circ$ and $OB^\circ$ on $[a, b]$, then $F + G \in OM^\circ$ on $[a, b]$.

Lemma 1.4 is proved in a previous paper by the author [5, Theorem 2]. In the original version [5, Theorem 2] the theorem is stated with the requirement that $\int_a^b G$ exist rather than $G \in OM^\circ$ on $[a, b]$. However, Lemma 1.3 establishes the equivalence of the two forms.
Lemma 1.5. If $E$ is a finite set of points from $[a, b]$ and $F$, $G$ and $H$ are functions from $R \times R$ to $R$ such that

1. $G \in OB^\circ$ and $S_2$ on $[a, b]$,
2. $H \in OP^\circ$ and $S_1 \cap S_2$ on $[a, b]$,
3. $H+G \in OM^\circ$ on $[a, b]$ and
4. $F \in S_2$ on $[a, b]$ and if $a \leq x < y \leq b$, then $F(x, y) = H(x, y)$ if $x \notin E$ and $y \notin E$,

then $F+G \in OM^\circ$ on $[a, b]$.

Proof. Lemma 1.1 establishes that $H+G \in OP^\circ$ on $[a, b]$. Further, $H+G \in S_1 \cap S_2$ on $[a, b]$. Let $H'$ be the function defined on $[a, b]$ such that

1. $H'(x, y) = 0$ if $x \notin E$ and $y \notin E$, and
2. $H'(x, y) = F(x, y) - H(x, y)$ if $x \in E$ or $y \in E$.

Thus, $H' \in OM^\circ$ and $OB^\circ$ on $[a, b]$. Hence, since $H+G+H' \equiv F+G$ on $[a, b]$,

$F+G \in OM^\circ$ on $[a, b]$.

Lemma 1.6. If $G$ is a bounded function from $R \times R$ to $R$ such that $\alpha \prod_b (1+G)$ exists and is not zero, then $G \in OP^\circ$ and $OQ^\circ$ on $[a, b]$.

Lemma 1.6 is a special case of a previous result by the author [4, Theorem 2].

Lemma 1.7. If $G$ is a bounded function from $R \times R$ to $R$ such that $G \in OM^\circ$ on $[a, b]$ and $1+G$ is bounded away from zero on $[a, b]$, then $G \in OP^\circ$ and $OQ^\circ$ on $[a, b]$.

Proof. Since $G \in OM^\circ$ on $[a, b]$ and $1+G$ is bounded away from zero on $[a, b]$, $\alpha \prod_b (1+G)$ exists and is not zero. Therefore, it follows from Lemma 1.6 that $G \in OP^\circ$ and $OQ^\circ$ on $[a, b]$.

Lemma 1.8. If $G$ is a function from $R \times R$ to $R$ such that $G \in OB^\circ$ on $[a, b]$ and $1+G$ is bounded away from zero on $[a, b]$, then $G \in OQ^\circ$ on $[a, b]$.

Lemma 1.8 is a special case of a previous result by the author [5, Theorem 3].

Lemma 1.9. If $G$ is a bounded function from $R \times R$ to $R$ such that $\alpha \prod_b (1+G)$ exists and is not zero, then $G \in OM^\circ$ on $[a, b]$.

Lemma 1.9 follows from Lemma 1.6 and a result of B. W. Helton [2, Theorem 4.2, p. 305].

Lemma 1.10. If $G$ is a function from $R \times R$ to $R$ such that $G \in OB^\circ$ and $S_2$ on $[a, b]$ and $\alpha \prod_y (1+G)$ exists for $a \leq x < y \leq b$, then $G \in OM^\circ$ on $[a, b]$.
Proof. Let $\varepsilon > 0$. Since $G \in OB^c$ on $[a, b]$, there exist a subdivision $D_0$ of $[a, b]$ and a number $B > 1$ such that if $\{x_i\}_{i=0}^n$ is a refinement of $D_0$ and $1 \leq r \leq s \leq n$, then

$$\left| \prod_{i=r}^{s} (1 + G_i) \right| < B.$$ 

There exists a subdivision $E = \{w_q\}_{q=0}^t$ of $[a, b]$ such that if $1 \leq q \leq t$ and $w_{q-1} < x < y < w_q$, then $|G(x, y)| < \frac{1}{2}$. Further, there exist sequences $\{u_q\}_{q=1}^t$ and $\{v_q\}_{q=1}^t$ such that

(1) $w_{q-1} < u_q < v_q < w_q$,
(2) if $w_{q-1} < x < y \leq u_q$, then $|G(w_{q-1}, x) - G(w_{q-1}, y)| < \varepsilon (8t)^{-1}$,
(3) if $w_{q-1} < x < u_q$ and $J$ is a subdivision of $[x, u_q]$, then $\sum_{J(t)} |G| < \varepsilon (8B^2t)^{-1}$,
(4) if $u_q \leq x < y < w_q$, then $|G(x, w_q) - G(y, w_q)| < \varepsilon (8t)^{-1}$,
and
(5) if $v_q < x < w_q$ and $J$ is a subdivision of $[v_q, x]$, then $\sum_{J(t)} |G| < \varepsilon (8B^2t)^{-1}$.

We know from the hypothesis that $\prod_{q=0}^t (1 + G)$ exists for $1 \leq q \leq t$. Further, it follows from Lemma 1.8 that each of these integrals is nonzero. Thus, Lemma 1.9 implies that $G \in OM^o$ on $[u_q, v_q]$. Hence, for $1 \leq q \leq t$, there exists a subdivision $D_q$ of $[u_q, v_q]$ such that if $J = \{x_i\}_{i=0}^n$ is a refinement of $D_q$ and $\{x_{ij}\}_{j=0}^n$ is a subdivision of $[x_{i-1}, x_i]$, then

$$\sum_{i=1}^n \left| 1 + G_i - \prod_{j=1}^{n(i)} (1 + G_{ij}) \right| < \varepsilon (8t)^{-1}.$$ 

Let $D$ denote the subdivision $\bigcup_{q=0}^t D_q \cup E$ of $[a, b]$. Suppose $\{x_{ij}\}_{i=0}^n$ is a refinement of $D$. Let $\{x_{w(i)}\}_{i=0}^n$ be the subsequence of $\{x_i\}_{i=0}^n$ such that $x_{w(i)} = w_i$. Further, let $\{x_{u(i)}\}_{i=1}^n$ and $\{x_{v(i)}\}_{i=1}^n$ be the subsequences of $\{x_i\}_{i=0}^n$ such that $x_{u(i)} = u_i$ and $x_{v(i)} = v_i$. Let $T(q)$, $U(q)$ and $V(q)$ denote $\{i_{w(q)}\}_{i=0}^{w(q)}$, $\{i_{w(q)}\}_{i=0}^{u(q)}$ and $\{i_{w(q)}\}_{i=0}^{v(q)}$, respectively. Further, let $U$, $V$, $U'(q)$ and $V'(q)$ denote $\{w(i)\}_{i=0}^n$, $\{w(i)\}_{i=0}^n$, $\{i_{w(q)}\}_{i=0}^{w(q)-1}$ and $\{i_{w(q)}\}_{i=0}^{w(q)-1}$, respectively. Finally, let $S(q)$ and $S'(q)$ represent $U(q) \cup V(q)$ and $U'(q) \cup V'(q)$, respectively.
For $1 \leq i \leq n$, there exists a subdivision $(x_{ij})_{j=0}^{n(i)}$ of $[x_{i-1}, x_i]$ such that

$$\left| \prod_{j=1}^{n(i)} (1 + G_{ij}) - x_{i-1} \prod_{j=1}^{n(i)} (1 + G) \right| < \varepsilon(8n)^{-1}.$$ 

Let $H_i$ represent $1 + G_i - \prod_{j=1}^{n(i)} (1 + G_{ij})$. Thus,

$$\sum_{i=1}^{n} \left| 1 + G_i - \prod_{j=1}^{n(i)} (1 + G) \right| < \sum_{i=1}^{n} |H_i| + \varepsilon(n)^{-1}n.$$

$$= \sum_{i=1}^{n} \left( \sum_{q=1}^{t} \sum_{e \in S(e)} |H_i| + \varepsilon \right)$$

$$< \sum_{i=1}^{n} \left( |H_i| + \varepsilon(n)^{-1}t \right) + \varepsilon$$

$$= \sum_{i=1}^{n} |H_i| + \sum_{i=1}^{n} \sum_{q=1}^{t} |H_i| + \frac{\varepsilon}{4}$$

$$= \sum_{i=1}^{n} |H_i| + \frac{\varepsilon}{4}$$

$$\leq \sum_{i=1}^{n} |H_i| + \frac{\varepsilon}{4}$$

$$\leq \sum_{i=1}^{n} \left| (1 + G_i) - (1 + G_{i+1}) \right| + \sum_{i=1}^{n} |1 + G_i| \left| -1 + \prod_{j=1}^{n(i)} (1 + G_{ij}) \right|$$

$$+ \sum_{i=1}^{n} \left| (1 + G_i) - (1 + G_{i,n(i)}) \right|$$

$$+ \sum_{i=1}^{n} \left| 1 + G_{i,n(i)} \right| \left| -1 + \prod_{j=1}^{n(i)-1} (1 + G_{ij}) \right| + \frac{\varepsilon}{2}$$

$$\leq \sum_{i=1}^{n} |G_i - G_{i+1}| + B \sum_{i=1}^{n} \left| -1 + \prod_{j=2}^{n(i)-1} (1 + G_{ij}) \right| [G_{ij}]$$

$$+ \left( \prod_{j=1}^{n(i)} (1 + G_{ij}) \right) \left[ \prod_{j=n(i)+1}^{n(i)-1} (1 + G_{ji}) \right]$$

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\[ + \sum_{i \in V} |G_i - G_{i,n(i)}| + B \sum_{i \in V} \left| -1 + \left\{ \sum_{j=1}^{n(i)-1} \left[ \prod_{r=1}^{j-1} (1 + G_{ir}) \right] [G_{ij}] \right\} \cdot \left[ \prod_{r=j+1}^{n(i)-1} (1 + G_{ir}) \right] \right| + \frac{\varepsilon}{2} \]

\[ < [\varepsilon(8t)^{-1}] t + B^2 \sum_{i \in V} \sum_{j=1}^{n(i)} |G_{ij}| \]

\[ + [\varepsilon(8t)^{-1}] t + B^2 \sum_{i \in V} \sum_{j=1}^{n(i)-1} |G_{ij}| + \frac{\varepsilon}{2} \]

\[ < B^3 [\varepsilon(8B^3t)^{-1}] t + B^2 [\varepsilon(8B^3t)^{-1}] t + \frac{3\varepsilon}{4} \]

\[ = \varepsilon. \]

Therefore, \( G \in OM^0 \) on \([a, b]\).

Lemma 1.10 is not true if only \( \int f^* (1+G) \) is required to exist rather than \( \int f^* (1+G) \) for \( a \leq x < y \leq b \). For example, consider the function \( G \) defined on \([0, 1]\) such that, for \( 0 \leq x < y \leq 1 \),

1. \( G(0, x) = -1 \),
2. \( G(x, y) = y - x \) if \( x \neq 0 \) and \( y \) is irrational, and
3. \( G(x, y) = x - y \) if \( x \neq 0 \) and \( y \) is rational.

Thus, \( G \in OB^0 \) and \( S_2 \) on \([a, b]\) and \( \int g \) \( (1+G) \) does not exist for \( a \leq x < y \leq b \), and thus, \( G \notin OM^0 \) on \([a, b]\).

**Lemma 1.11.** If \( H \) and \( G \) are functions from \( R \times R \) to \( R \) such that \( H \in OL^0 \) on \([a, b]\), \( G \in OB^0 \) on \([a, b]\) and either \( G \in OM^0 \) on \([a, b]\) or \( G \in OA^0 \) on \([a, b]\), then \( HG \in OM^0 \) and \( OA^0 \) on \([a, b]\).

Lemma 1.11 is a modification of a result of B. W. Helton [3, Theorem 2, p. 494] obtained by using Lemma 1.3.

**Theorem 1.** If \( F \) and \( G \) are functions from \( R \times R \) to \( R \) such that \( F \in OP^0 \) and \( S_1 \cap S_2 \) on \([a, b]\) and \( G \in OB^0 \) and \( S_2 \) on \([a, b]\), then any two of the following statements imply the other:

1. \( F + G \in OM^0 \) on \([a, b]\),
2. \( F \in OM^0 \) on \([a, b]\), and
3. \( G \in OM^0 \) on \([a, b]\).

**Proof (1, 2 → 3).** There exists a subdivision \( E = \{w_i\}_{i=0}^{t} \) of \([a, b]\) such that if \( 1 \leq i \leq t \) and \( w_{i-1} < x < y < w_i \), then \( |F(x, y)| < \frac{1}{4} \). Let \( F'(x, y) = F(x, y) \) if \( x \not\in E \) and \( y \not\in E \), and let \( F'(x, y) \) = 0 if \( x \in E \) or \( y \in E \). Thus, \((1+F')^{-1} \) is in \( OL^0 \) on \([a, b]\). Further, it follows from Lemma 1.5 that \( F' + G \in OM^0 \) on \([a, b]\) and \( F' \in OM^0 \) on \([a, b]\). Hence, since \( F' \in OM^0 \) on
Lemma 1.7 implies that $F' \in OQ^c$ on $[a, b]$. Also, note that $G(1+F')^{-1}$ is in $OB^c$ and $OP^c$ on $[a, b]$.

We now establish that $\prod_{a \leq x < y \leq b} [1+G(1+F')^{-1}]$ exists by using the Cauchy criterion for product integrals, where $a \leq x < y \leq b$. Let $\varepsilon > 0$. There exist a subdivision $D$ of $[x, y]$ and positive numbers $B$ and $\beta$ such that if $J$ and $K$ are refinements of $D$, then

1. $|\prod_{J} (1+F')| > \beta$,
2. $|\prod_{J} [1+G(1+F')^{-1}]| < B$,
3. $|\prod_{J} (1+F') - \prod_{K} (1+F')| < \beta \varepsilon (2B)^{-1}$, and
4. $|\prod_{J} (1+F'+G) - \prod_{K} (1+F'+G)| < \beta \varepsilon /2$.

Suppose $J$ and $K$ are refinements of $D$. Thus,

$$\frac{\beta \varepsilon}{2} > \left| \prod_{J} (1+F'+G) - \prod_{K} (1+F'+G) \right|$$

$$= \left| \left( \prod_{J} (1+F') \right) \left( \prod_{J} [1+G(1+F')^{-1}] \right) - \left( \prod_{K} (1+F') \right) \left( \prod_{K} [1+G(1+F')^{-1}] \right) \right|$$

$$\geq \left| \prod_{J} (1+F') \right| \left| \prod_{J} [1+G(1+F')^{-1}] - \prod_{K} [1+G(1+F')^{-1}] \right|$$

$$- \left| \prod_{J} (1+F') - \prod_{K} (1+F') \right| \left| \prod_{K} [1+G(1+F')^{-1}] \right|$$

$$\geq \beta \left| \prod_{J} [1+G(1+F')^{-1}] - \prod_{K} [1+G(1+F')^{-1}] \right| - [\beta \varepsilon (2B)^{-1}]B,$$

and hence,

$$\varepsilon > \left| \prod_{J} [1+G(1+F')^{-1}] - \prod_{K} [1+G(1+F')^{-1}] \right|.$$

Therefore, the desired product integral exists.

Now, since $\prod_{a \leq x < y \leq b} [1+G(1+F')^{-1}]$ exists for $a \leq x < y \leq b$ and $G(1+F')^{-1}$ is in $OB^c$ on $[a, b]$, it follows from Lemma 1.10 that $G(1+F')^{-1}$ is in $OM^c$ on $[a, b]$. Hence, since $1+F'$ is in $OL^c$, Lemma 1.11 implies that $G \in OM^c$ on $[a, b]$.

**Proof (2, 3→1).** This result is stated as Lemma 1.4 and is proved in a previous paper by the author [5, Theorem 2].

**Proof (1, 3→2).** It follows from Lemma 1.1 that $F+G \in OP^c$ on $[a, b]$. Further, $F+G \in S_1 \cap S_2$ on $[a, b]$, and $-G \in OB^c$ and $OM^c$ on $[a, b]$. Therefore, Lemma 1.4 implies that $F \equiv F+G-G$ is in $OM^c$ on $[a, b]$. 

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DEPARTMENT OF MATHEMATICS, ARIZONA STATE UNIVERSITY, TEMPE, ARIZONA 85281