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A NOTE ON ANALYTIC MEASURES

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ABSTRACT. A corollary to Forelli's generalization of the F. and M. Riesz theorem is proved. It extends Bochner's result concerning the absolute continuity of measures on the torus whose Fourier-Stieltjes coefficients vanish outside a sector of opening less than π.

In this note, X will be a left coset space G/H where G is a connected Lie group and H is a closed subgroup. Each vector V in the Lie algebra $\mathfrak{G}$ of G determines a one-parameter group $\{T_t\}_{t\in \boldsymbol{R}}$ of homeomorphisms on X via the formula $T_t(gH)=[\exp(tV)g]H$, $t\in \boldsymbol{R}$, $gH\in X$. We shall refer to $\{T_t\}_{t\in \boldsymbol{R}}$ as the *flow* determined by V. If $\{T_t\}_{t\in \boldsymbol{R}}$ is a flow on X and if μ belongs to $M(X)$, the space of finite regular Baire measures on X, then μ is called analytic with respect to $\{T_t\}_{t\in \boldsymbol{R}}$ in case the function of t, $\int_X \phi\circ T_{-t}\,d\mu$, lies in $H^\infty(\boldsymbol{R})$ for each ϕ in $C_0(X)$. A measure μ in $M(X)$ is called *quasi-invariant* with respect to $\{T_t\}_{t\in \boldsymbol{R}}$ in case $|\mu|$ and $|\mu|\circ T_t$ have the same null sets for each t in $\boldsymbol{R}$ ($|\mu|$=total variation measure of μ). In [3] Forelli proved that an analytic measure is quasi-invariant. In this note we prove a corollary to Forelli's theorem which extends Bochner's theorem [**1**, Theorem 5] and a theorem of de Leeuw and Glicksberg [**2**, Theorem 3.4].

THEOREM. *Let $V_1,\cdots,V_n$ be a basis for $\mathfrak{G}$ and let $\{T_t^{(i)}\}_{t\in \boldsymbol{R}}$ be the flow on X determined by V_i, $i=1,\cdots,n$. If μ is a nonzero measure in $M(X)$ which is analytic with respect to each flow $\{T_t^{(i)}\}_{t\in \boldsymbol{R}}$, then μ is equivalent to the transplant to X of (left or right) Haar measure on G.*

PROOF. By a theorem of Mackey [**4**, Theorem 1.1] it suffices to show that $|\mu|(gE)=0$ for each null set E for $|\mu|$ and for each g in G. For g in G,

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define T_g by the formula

$$T_g(xH) = (gx)H, \qquad xH \in X.$$

First we show that $|\mu| \circ T_g$ moves continuously through $M(X)$. Since the map $\lambda \to \lambda \circ T_g$, $g \in G$, $\lambda \in M(X)$ defines an isometric representation of G on $M(X)$, it suffices to check continuity at the identity e of G. By hypothesis and Theorem 4 in [3], $|\mu| \circ T_t^{(i)}$ moves continuously through $M(X)$ for each i. Hence, as a calculation reveals, the map

$$(t_1, \cdots, t_n) \to |\mu| \circ T_{t_1}^{(1)} \circ T_{t_2}^{(2)} \circ \cdots \circ T_{t_n}^{(n)}$$

is continuous from $\boldsymbol{R}^n$ into $M(X)$. By the inverse function theorem, there is a neighborhood N_0 of the origin in $\boldsymbol{R}^n$ and a neighborhood N_e of e in G such that the map

$$(t_1, \cdots, t_n) \to \exp(t_1 V_1)\exp(t_2 V_2) \cdots \exp(t_n V_n)$$

is a diffeomorphism from N_0 onto N_e. The continuity at e of the map $g \to |\mu| \circ T_g$ follows. Let E be a Baire set in X. Since $|\mu| \circ T_g$ moves continuously through $M(X)$, the set of g such that $|\mu| \circ T_g(E)=0$ is closed. Since μ is quasi-invariant with respect to each $\{T_t^{(i)}\}_{t \in \boldsymbol{R}}$, it follows that if $|\mu| \circ T_g(E)=0$ for some g in N_e, then $|\mu| \circ T_g(E)=0$ for all g in N_e. Hence, since $g \to |\mu| \circ T_g$ is continuous, the set of all g such that $|\mu| \circ T_g(E)=0$ is open. Since G is connected, this set is either all of G or empty, and the proof is complete.

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