Invariant Traces on Algebras

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Abstract. Certain properties of traces on a finite-dimensional associative algebra \( A \) lead to the definition of an element \( t(A) \in H^1(\text{Out } A, C^*) \), \( C^* \) being the multiplicative group of the center of \( A \) as \( \text{Out } A \)-module. It is shown that \( t(A) = 0 \) is equivalent to the existence of nondegenerate traces on \( A \) which are invariant under composition with all automorphisms of \( A \). In particular, by means of Galois theory, \( t(A) = 0 \) is shown for a semisimple algebra \( A \), whereas \( t(A) \neq 0 \) for certain group algebras.

1. Let \( R \) be a field, \( A \) an associative unitary algebra of finite dimension over \( R \). By a trace on \( A \) we mean a linear map \( \tau: A \rightarrow R \) such that \( \tau(ab) = \tau(ba) \) \( \forall a, b \in A \). This is one possible generalization of the notion of a trace on matrix rings (see [4]; for a generalization in another context, see [2]).

In §§2–4 we shall list some generalities on traces; let \( T(A) \) be the \( R \)-vectorspace of all traces on \( A \).

2. The existence of nonzero traces on \( A \) depends on the abelianized algebra \( A^a \). Let \( [A, A] \) be the vectorspace generated by all commutators \( [a, b] = ab - ba \) in \( A \), \( A^a \) the quotient \( A/[A, A] \). The class map \( \pi: A \rightarrow A^a \) provides an isomorphism of vectorspaces

\[ \pi^*: \text{Hom}_R(A^a, R) \rightarrow T(A), \]

where \( \pi^* \) is the dual map of \( \pi \).

One knows that \( A^a \neq (0) \) if \( A \) is simple [1], hence

\[ (2.1) \quad T(A) \neq (0) \quad \text{for a simple algebra } A. \]

3. The radical of a trace \( \tau \) is the set

\[ R_\tau = \{ a \in A / \tau(ab) = 0 \ \forall b \in A \}, \]

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and $\tau$ is nondegenerate if $R_\tau = (0)$. As $R_\tau$ is a 2-sided ideal, 

\[(3.1) \quad \text{a nonzero trace on a simple algebra is nondegenerate.}\]

$T(A)$ is a module over the center $C$ of $A$, as $z \cdot \tau$ for $z \in C$ and $\tau \in T(A)$ defined by 

\[(3.2) \quad (z \cdot \tau)(a) := \tau(za), \quad a \in A,
\]

is again a trace.

**Proposition 1.** A nondegenerate trace $\eta \in T(A)$ is a free generator of the $C$-module $T(A)$.

**Proof.** $\eta$ provides a linear isomorphism from $A$ to its dual $\text{Hom}_R(A, R)$; for every $\tau \in T(A) \subset \text{Hom}_R(A, R)$ there exists a unique $b \in A$ such that $\tau(a) = \eta(ba) \forall a \in A$. We then have the following sequence of implications

\[
\begin{align*}
\tau(a_1a_2) &= \tau(a_2a_1) \quad \forall a_i \in A \\
\Rightarrow \quad \eta(ba_1a_2) &= \eta(ba_2a_1) = \eta(a_1ba_2) \quad \forall a_i \in A \\
\Rightarrow \quad ba_1 - a_1b &\in R\eta = (0) \quad \forall a_1 \in A \\
\Rightarrow \quad b &\in C \Rightarrow \tau = b \cdot \eta.
\end{align*}
\]

**Corollary 1.** Suppose the set $B(A)$ of nondegenerate traces on $A$ is nonempty. Then, the $C$-module structure of $T(A)$ defines a simply transitive action of $C^\times$ on $B(A)$, where $C^\times$ is the multiplicative group of invertible elements of the center $C$ of $A$.

4. Let $\text{Aut} A, \text{In} A$ denote the group of all automorphisms and anti-automorphisms, of all inner automorphisms resp. of $A$, and denote the quotient group $\text{Aut} A/\text{In} A$ by $\text{Out} A$. As can be seen immediately from the definitions, composing an (anti-) automorphism with a trace yields again a trace and thus an operation of $\text{Aut} A$ on $T(A)$. Inner automorphisms act in this way as the identity, and we finally get an action of $\text{Out} A$ on $T(A)$. Let $\tau \cdot \omega$ ($\tau \in T(A), \omega \in \text{Out} A$) be the symbol for this action. Its relationship with the $C$-module structure of $T(A)$ may be described in the form of an associative law

\[(4.1) \quad (\omega c) \cdot (\tau \cdot \omega^{-1}) = (c \cdot \tau) \cdot \omega^{-1}, \quad c \in C, \tau \in T(A), \omega \in \text{Out} A.
\]

5. We say that a trace $\tau$ is invariant if $\tau \cdot \omega = \tau \forall \omega \in \text{Out} A$. We are coming now to the main point of this note which consists in giving a condition on the cohomology level for the existence of nondegenerate invariant traces.

In the subsequent statement, $C^\times$ is meant to be an $\text{Out} A$-module via the operation of automorphisms on the center.
Proposition 2. For every algebra $A$ with $B(A) \neq \emptyset$, there is defined an element $t(A) \in H^1(\text{Out } A, C^*)$ such that $t(A) = 0$ precisely if $A$ has nondegenerate invariant traces.

Proof. By Corollary 1, there belongs to every $\tau \in B(A)$ a map $f_\tau : \text{Out } A \to C^*$ such that
\begin{equation}
(5.1) \quad \tau \cdot \omega^{-1} = f_\tau(\omega) \cdot \tau \quad \forall \omega \in \text{Out } A.
\end{equation}
Then, the following statements are immediate consequences of (4.1):

(1) $f_\tau$ is a crossed homomorphism.
(2) For $\tau$ and $\eta \in B(A)$, $f_\tau$ and $f_\eta$ differ by a principal crossed homomorphism.

If $t(A)$ is then defined as the cohomology class of the $f_\tau$'s the statement in Proposition 2 on $t(A)$ is easily verified using again (4.1).

6. Example 1. $t(A) = 0$ for a semisimple algebra $A$. In fact, if $A$ is simple we know from (2.1) and (3.1) that $B(A) \neq \emptyset$. As Out $A$ is a finite group and $C^*$ the multiplicative group of a field, a fundamental theorem of Galois theory asserts that $H^1(\text{Out } A, C^*) = 0$ [3, Chapter IV, p. 106]. By Proposition 2, $A$ has nondegenerate invariant traces.

If $A = \bigoplus A_i$ ($1 \leq i \leq n$) is semisimple with simple components $A_i$, let
\begin{equation}
\text{Out } A := \{ \omega \in \text{Out } A / \omega(A_i) \subset A_i \}.
\end{equation}
Choose one index $i$ for each conjugacy class of the subgroups $\text{Out } i A \subset \text{Out } A$, and on $A_i$ a nondegenerate invariant trace $\tau_i$. If $\text{Out } i A$ is conjugate to $\text{Out } j A$, there exists $\alpha \in \text{Aut } A$ with $\alpha : A_k \to A_i$, and define $\tau_k$ on $A_k$ by $\tau_k = \tau_i \circ \alpha$. The direct sum of all these traces on the different $A_i$ is seen to be a nondegenerate invariant trace on $A$.

7. Example 2. Let $G_p$ be a finite cyclic group of prime order $p > 2$, $R = \mathbb{Z}_p$ and $A$ the group algebra $\mathbb{Z}_p(G_p)$. Then, $t(A) \neq 0$.

First we note, that in the more general situation of a finite group $G$ and field $R$, the group algebra $R(G)$ has at least one nondegenerate trace $\tau_0$ given by $\tau_0(x) = x(1)$ where $x = \sum x(g) \cdot g \in R(G)$, $g \in G$ and $x(g) \in R$, and $1$ is the unit in $G$. Hence, $t(R(G))$ is defined.

Suppose now $q$ is a generator of $G_p$. As $A = \mathbb{Z}_p(G_p)$ is commutative, we have $\text{Out } A = \text{Aut } A$ and every $\alpha \in \text{Aut } A$ is characterized by its value on $q$. If $x = \alpha(q)$, $x^p = 1$ and the powers $x^r$ for $1 \leq r \leq p - 1$, form an $R$-basis of $A$. Conversely, every $x \in A$ with this property is the value of some $\alpha \in \text{Aut } A$ on $q$. Therefore at least $p$ automorphisms $\alpha_v$, $0 \leq v \leq p - 1$, of $A$ exist which are given by their values on $q$: $\alpha_v(q) = q^r$, $1 \leq r \leq p - 1$, $\alpha_0(q) = \frac{1}{2}(1 + q)$. 
From this we conclude that an invariant trace $\tau$ on $A = \mathbb{Z}_p(G_p)$ must assume the same value on each $q^r$, $0 \leq r \leq p - 1$, and as such must be a multiple of the augmentation $\varepsilon: \mathbb{Z}_p(G_p) \rightarrow \mathbb{Z}_p$. The kernel of $\varepsilon$ being an ideal, $\varepsilon$ is a degenerate trace and so is $\tau$.

**References**


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