A NOTE ON MONOMIALS IN SEVERAL COMPLEX VARIABLES

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Abstract. Monomials in \( C^n \) are characterized in the polydisk algebra \( A(U^n) \) as functions whose modulus is constant on the distinguished boundary of \( U^n \) and whose zero set has an intersection with the diagonal of \( U^n \) consisting (at most) of the origin.

The note answers a question raised by F. Norguet in his course notes Fonctions de plusieurs variables complexes, Paris, 1971. The following characterization of monomials in the polydisk algebra is a generalization of a result of R. Bojanic and W. Stoll about a characterization of monomials among entire functions [1].

We use the following notations:
- \( U^n \) is the open unit polydisk in \( C^n \).
- \( T^n \) is the distinguished boundary of \( U^n \).
- \( \Delta^n = \{ (\lambda, \cdots, \lambda) \in \overline{U}^n, \lambda \in \overline{U}^1 \} \) is the diagonal of \( \overline{U}^n \).
- \( A(U^n) \) is the algebra of functions analytic in \( U^n \) and continuous in \( \overline{U}^n \).

For \( f \in A(U^n) \) we call \( Z_f \), the zero set of \( f \) in \( \overline{U}^n \).

Theorem. Assume that \( f \in A(U^n) \) satisfies

(i) \( |f(T^n)| = 1 \),
(ii) \( Z_f \cap \Delta^n = \{ (0, \cdots, 0) \} \).

Then \( f = cz_1^{k_1} \cdots z_n^{k_n} \) where \( c \in T^1 \).

Proof. Let \( z \in T^n \). Then \( f_s(\lambda) = f(\lambda z) \) is a finite Blaschke product in \( A(U^1) \) since (i)\(\Rightarrow \) \( f \) is an “inner function” in \( A(U^n) \), hence a rational function [2, p. 112]. Hence \( f_s(\lambda) = c(z)\lambda^{p(z)} \) by (ii), where \( p(z) \) is nonnegative integer valued. For \( \lambda = 1 \), \( f(z) = c(z) \forall z \in T^n \), hence \( f_s(\lambda) = f(z)\lambda^{p(z)} \). This shows that \( p(z) \) is continuous integer valued on \( T^n \) for fixed \( \lambda \in \overline{U}^1 \). Hence \( p(z) = k \), a fixed nonnegative integer since \( T^n \) is connected.

It follows that \( f(\lambda z) = \lambda^k f(z) \forall z \in T^n \) hence also for \( \forall z \in U^n \). As seen from the Taylor expansion of \( f \), \( f \) must be a homogeneous polynomial.
of degree \( k \). (This part of the proof parallels an argument of S. Bochner [3].) Denote by \( k_i \) the degree of \( f \) in \( z_i \) \((i=1, \ldots, n)\). Write \( f(z) = Q_i(z)z_i^{k_i} \) plus terms of lower degree in \( z_i \) where \( Q_i \neq 0 \) is a polynomial that does not involve \( z_i \). There is at least one point of \( T^n \) where all \( Q_i \) are \( \neq 0 \). (If some \( Q_i \) were zero at each point of \( T^n \), then \( \prod_i Q_i \) would be zero on \( T^n \) and hence identically zero on \( U^n \). Hence one of the \( Q_i \)'s would be identically zero—a contradiction.)

Assume such a point is \((1, 1, \ldots, 1)\); now \( f(\lambda, 1, 1, \ldots, 1) \) is a Blaschke product hence it has all its \( k_\lambda \) zeros in \( U^1 \). An easy index computation (see [2, p. 89]) shows that \( f(\lambda, \lambda, \ldots, \lambda) \) has degree \( k_1 + \cdots + k_n \). It follows that \( f \) is a monomial, \( f = c z_1^{k_1} \cdots z_n^{k_n} \) where \( c \in T^1 \).

BIBLIOGRAPHY


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