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A SHORT PROOF OF VAN DER WAERDEN'S THEOREM ON ARITHMETIC PROGRESSIONS

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ABSTRACT. A short proof is given for the classical theorem of van der Waerden which asserts that for any partition of the integers into a finite number of classes, some class contains arbitrarily long arithmetic progressions.

Let $[a, b]$ denote the set of integers x with $a \leqq x \leqq b$. We call $(x_1, \cdots, x_m)$, $(x'_1, \cdots, x'_m) \in [0, l]^m$ *l-equivalent* if they agree up through their last occurrences of l. For any l, $m \geqq 1$, consider the statement

$S(l, m)$ For any r, there exists $N(l, m, r)$ so that for any function $C: [1, N(l, m, r)] \to [1, r]$, there exist positive $a, d_1, \cdots, d_m$ such that $C(a+\sum_{i=1}^m x_i d_i)$ is constant on each l-equivalence class of $[0, l]^m$.

FACT 1. *$S(l, m)$ for some $m \geqq 1 \Rightarrow S(l, m+1)$.*

PROOF. For a fixed r, let $M=N(l, m, r)$, $M'=N(l, 1, r^M)$ and suppose $C: [1, MM'] \to [1, r]$ is given. Define $C': [1, M'] \to [1, r^M]$ so that $C'(k)=C'(k')$ iff $C(kM-j)=C(k'M-j)$ for all $0 \leqq j < M$. By the inductive hypothesis, there exist a' and d' such that $C'(a'+xd')$ is constant for $x \in [0, l-1]$. Since $S(l, m)$ can apply to the interval $[a'M+1, (a'+1)M]$, then by the choice of M, there exist $a, d_1, \cdots, d_m$ with all sums $a+\sum_{i=1}^m x_i d_i$, $x_i \in [0, l]$, in $[a'M+1, (a'+1)M]$ and with $C(a+\sum_{i=1}^m x_i d_i)$ constant on l-equivalence classes. Set $d'_i=d_i$ for $i \in [1, m]$ and $d'_{m+1}=d'M$; then $S(l, m+1)$ holds.

FACT 2. *$S(l, m)$ for all $m \geqq 1 \Rightarrow S(l+1, 1)$.*

PROOF. For a fixed r, let $C: [1, 2N(l, r, r)] \to [1, r]$ be given. Then there exist $a, d_1, \cdots, d_r$ such that for $x_i \in [0, l]$, $a+\sum_{i=1}^r x_i d_i \leqq N(l, r, r)$ and $C(a+\sum_{i=1}^r x_i d_i)$ is constant on l-equivalence classes. By the box principle there exist $u < v$ in $[0, r]$ such that

$$C\left(a+\sum_{i=1}^{u} l d_i\right) = C\left(a+\sum_{i=1}^{v} l d_i\right).$$

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Therefore $C((a+\sum_{i=1}^{u} ld_i)+x(\sum_{i=u+1}^{v} d_i))$ is constant for $x \in [0, l]$. This proves $S(l+1, 1)$.

Since $S(1, 1)$ holds trivially, then by induction $S(l, m)$ is valid for all $l, m \geqq 1$. Van der Waerden's theorem is $S(l, 1)$.

The authors point out that while previous proofs follow essentially the argument above, the one given is hopefully clearer.

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