ASYMPTOTICS OF STIRLING NUMBERS OF THE SECOND KIND

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ABSTRACT. A complete asymptotic development of the Stirling numbers $S(N, K)$ of the second kind is obtained by the saddle point method previously employed by Moser and Wyman [Trans, Roy. Soc. Canad., 49 (1955), 49-54] and de Bruijn [Asymptotic methods in analysis, North-Holland, Amsterdam, 1958, pp. 102-109] for the asymptotic representation of the related Bell numbers.

1. Introduction. Hsu [1] has given the asymptotic expansion

$$S(N, K) \sim \left(\frac{1}{4}K^2\right)^{N-K}\left[1 + \sum_{i=1}^{K} \frac{K^{-i}f_i(N-K)}{(N-K)!} + O(K^{-i-1})\right]/(N-K)!$$

for Stirling numbers $S(N, K)$ of the second kind, where $f_i(N-K)$ are polynomials of argument $N-K$ and $f_i(0)=0$. The expansion (1) is useful only for $N-K$ small, as is indicated in §3. We obtain a complete asymptotic expansion of $S(N, K)$ in powers of $(N+1)^{-1}$, using the saddle point method previously employed by Moser and Wyman [2] and de Bruijn [3] for the asymptotic representation of the related Bell numbers. Convergence is demonstrated for $K < (N+1)^{3/2} / \pi + (N+1)^{-1/3}$. The expansion when divergent is still useful when used as an asymptotic series.

2. Asymptotics of $S(N, K)$. A generating function for $S(N, K)$ is

$$\left(\frac{e^z - 1}{z}\right)^K = \sum_{N=K}^{\infty} \frac{K!}{N!} S(N, K)z^{N-K}.$$ 

Hence the Cauchy integral formula gives

$$S(N, K) = \frac{N!}{2\pi i K!} \int_C (e^z - 1)^K z^{-N-1} \, dz$$


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where the contour $C$ encloses the origin. Equating the derivative of the integrand to zero gives the equation

$$
(t - z)e^{z-t} = te^{-t},
$$

where $t = (N+1)/K$, for the location of the saddle point of the modulus of the integrand. The principal saddle point $z = u$ is on the positive real axis with $t - 1 < u < t$. The quadratic approximation to $xe^{-x}$ at $x = 1$ shows that $u \approx 2/N$ for $K = N$ and large $N$. There are other subsidiary saddle points at complex roots of (4), which we neglect in comparison with the higher saddle point at $z = u$. Since there are no other roots of (4) for $|t - z| \leq t - u$, we may apply the Lagrange inversion formula to obtain

$$
u = t - \sum_{m=1}^{\infty} m^{m-1}(te^{-t})^{m}/m!
$$

convergent for $t > 1$. Another form of (4) is the identity

$$K = (N + 1)(1 - e^{-u})/u
$$

needed later. Since the axis of the saddle point is perpendicular to the real axis, the part of the contour $C$ descending from $z = u$ is taken as the line $z = u + iy$, $|y| < \infty$, parallel to the imaginary axis. The modulus of the integrand in (3) is maximal at $z = u$ on this path, since both $(e^{z}-1)^K$ and $z^{-N-1}$ have this property. The closed contour $C$ is completed by a half circle of infinite radius enclosing the origin. The contribution to the integral (3) on this semicircular path is zero since $N > 0$. The integral in (3) then becomes

$$i(e^{u} - 1)^K u^{-N-1} \int_{-\infty}^{\infty} \exp \psi(u + iy) \, dy
$$

where

$$
\psi(z) = K \ln[(e^{z} - 1)/(e^{u} - 1)] - (N + 1)\ln(z/u).
$$

The contribution of the various parts of the $z = u + iy$ path to the integral must now be studied. As $|\exp \psi(z)| = \exp \Re \psi(z)$ we have to study

$$
\Re \psi(u + iy) = K \ln[(e^{2u} - 2e^u \cos y + 1)^{1/2}/(e^{u} - 1)] - (N + 1)\ln(1 + y^2u^{-2})^{1/2}.
$$

We shall show that we can restrict ourselves essentially to the interval $|y| < \pi$. Since $1 + y^2u^{-2} \geq 1 + \pi(y - \pi)u^{-2}$ for $y \geq \pi$ we have

$$(e^{u} - 1)^K u^{-N-1} \int_{-\infty}^{\infty} \exp \psi(u + iy) \, dy < \frac{u^{1-N}(e^{u} + 1)^K}{\pi(N - 1)(1 + \pi^2u^{-2})^{N/2-1/2}}.$$
which is of $O(N^{-N}e^N)$ for $K$ small and of $O(2^N/\pi^NN)$ for $K$ large. Since
$\Re \psi(u+iy)$ is even, the part of the integral (7) for $|y|>\pi$ tends toward
zero as $N\to\infty$. We now direct our attention to the interval $|y|<\pi$ where
the saddle point at $y=0$ gives the main contribution. The Taylor expansion
of $\psi(u+iy)$, convergent for $|y|<u$, is

$$\psi = -\frac{N+1}{2u}\left(\frac{1}{u} - \frac{1}{e^u - 1}\right) y^2$$

(9)

$$+ (N+1) \sum_{j=1}^{\infty} \frac{(iy)^{j+2}}{(j+2)!} \left(\frac{d}{dz}\right)^{j+1} \left\{ \frac{1}{u(e^z - 1)} - \frac{1}{z}\right\}_{z=u}$$

where the identity (6) has been used. We now make the substitutions

$$w = [(N+1)/2]^{1/2}[1 - u/(e^u - 1)]^{1/2}y/u$$

and

$$a_j = \frac{(iw)_{j+2}(d/dz)^{j+1}[(1 - e^{-z})/u(l^z - 1) - 1/z]_{z=u}}{(j+2)! \left[\frac{1}{2} - \frac{1}{2}u/(e^u - 1)\right]^{j+1}}$$

(11)

to obtain

$$S(N, K) = B \int_{-\infty}^{\infty} \exp\{-w^2 + f[(N + 1)^{-1/2}]\} \, dw$$

(12)

where

$$B = \frac{N!(e^u - 1)^K/\pi(2(N+1))^{1/2}K!}{u(N+1)(1 + u/(1 -\exp u))^{1/2}}$$

and $f$ is the analytic continuation of

$$f[(N + 1)^{-1/2}] = \sum_{j=1}^{\infty} a_j(N + 1)^{-j/2}.$$

(13)

To find an upper bound to $|a_j|$ we note that $(e^{z+1} - 1)^{-1} = \sum_{k=1}^{\infty} e^{-kz}$ for
$\Re z > 0$. Then

$$(d/du)^n(u e^{-u} - 1)^{-1} = (-1)^n \sum_{k=0}^{\infty} g(x)$$

where $g(x) = x^n e^{-ux}$. On using the Euler-Maclaurin sum formula [5] we find

$$\sum_{x=0}^{\infty} g(x) = \frac{n!}{u^{n+1}} + R_n.$$

The remainder $R_n = \int_0^\infty (x - [x]-\frac{1}{2})g'(x) \, dx$ may be evaluated by a Laplace
transform [6] to be

$$R_n = (-1)^{n-1} [nF^{(n-1)}(u) + uF^{(n)}(u)]$$
where $F(u) = u^{-2} - \frac{1}{2} u^{-1} \coth \frac{1}{2} u$. We conclude that $|R_u| \ll n!/u^{n+1}$ for small $u$ and tends to zero for large $u$. Since $(1 - e^{-u})/u$ is less than unity we have

$$|a_i| < \sigma^{i+2}/j$$

where

$$\sigma = w^{21/2}/(1 - u/(e^u - 1))^{1/2} = \gamma(N + 1)^{1/2}/u.$$  

Remembering that we need not integrate (7) beyond $|y| = \pi$ for large $N$, we see by (15) and (16) that the series (14) is convergent for $\pi/u < 1$. We now expand $\exp f[(N+1)^{-1/2}]$ in a Taylor series of the form

$$\exp f[(N + 1)^{-1/2}] = \sum_{j=0}^{\infty} b_j(N + 1)^{-j/2}$$

where $b_0 = 1$ and $b_j$ are polynomials in $w$ of the degree and parity of $3j$. By a lemma of Moser and Wyman [4]

$$|b_j| \leq \sigma^{j+2}(1 + \sigma^2)^{j-1}.$$  

Using (17) we may write (12) in the form

$$S(N, K) = B \left[ \sum_{j=0}^{\infty} (N + 1)^{-j} \int_{-\infty}^{\infty} e^{-w^2} b_{2j} \, dw + R_s \right].$$

The absolute value of the remainder $R_s$ is found from (18) to be

$$|R_s| \leq (N + 1)^{-j} \int_{-\infty}^{\infty} e^{-w^2} P_s(|w|) \, dw/M$$

where $P_s(|w|)$ is a polynomial in $|w|$ and

$$M = 1 - \sigma^2(1 + \sigma^2)^2/(N + 1).$$

On limiting the integration in (7) to $|y| < \pi$ we see that the remainder $R_s$ exists if

$$(\pi/u)[1 + (N + 1)(\pi/u)^2] < 1.$$  

Since $u+1>(N+1)/K$ convergence occurs for

$$K < (N + 1)^{2/3}/[\pi + (N + 1)^{-1/3}]$$

approximately. For these values of $K$ we conclude that

$$S(N, K) \sim B \left[ \sum_{j=0}^{\infty} (N + 1)^{-j} \int_{-\infty}^{\infty} e^{-w^2} b_{2j} \, dw + O((N + 1)^{-j}) \right].$$
The first two terms of (20) have been calculated to be

\[ S(N, K) \sim \frac{N!(e^u - 1)^K}{(2\pi(N+1))^{1/2}K!u^N(1-G)^{1/2}} \]

\[ \cdot \left[ 1 - \frac{2 + 18G - 20G^2(e^u + 1)}{24(N+1)(1-G)^3} \right. \]

\[ + \frac{3G^3(e^{2u} + 4e^u + 1) + 2G^4(e^{2u} - e^u + 1)}{24(N+1)(1-G)^3} \]

where by [5]

\[ G = u/(e^u - 1) = 1 - \frac{1}{2}u + \sum_{k=1}^{\infty} B_{2k}u^{2k}/(2k)! \]

The bracketed expression in (21), argumented by an additional inverse power of \( N+1 \), is approximated by

\[ 1 - \frac{1}{6u(N+1)} + \frac{1}{72u^3(N+1)^2} \]

for small \( u \) and by

\[ 1 - \frac{1}{12(N+1)} + \frac{1}{288(N+1)^2} \]

for large \( u \). These are the leading terms of an alternating asymptotic series.

3. Numerical example. The 6-significant-figure Table 1 compares the exact values of \( S(100, K) \) with the values computed from (20) and (1) for several \( K \). The excellent results obtained from (20) for values of \( K \) outside the interval of convergence show that the expansion gives useful results when used as an asymptotic series.

<table>
<thead>
<tr>
<th>( K )</th>
<th>( S(100, K) ) Exact</th>
<th>( S(100, K) ) 1 term of (20)</th>
<th>( S(100, K) ) 2 terms of (20)</th>
<th>( S(100, K) ) 4 terms of (1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6.33825 ( 10^9 )</td>
<td>6.34348 ( 10^9 )</td>
<td>6.33825 ( 10^9 )</td>
<td>1.81186 ( 10^{-11} )</td>
</tr>
<tr>
<td>25</td>
<td>2.58230 ( 10^{14} )</td>
<td>2.58496 ( 10^{14} )</td>
<td>2.58321 ( 10^{14} )</td>
<td>5.1529 ( 10^4 )</td>
</tr>
<tr>
<td>50</td>
<td>4.30983 ( 10^{10} )</td>
<td>4.30990 ( 10^{10} )</td>
<td>4.30977 ( 10^{10} )</td>
<td>1.51529 ( 10^4 )</td>
</tr>
<tr>
<td>75</td>
<td>1.82584 ( 10^{8} )</td>
<td>1.82671 ( 10^{8} )</td>
<td>1.82579 ( 10^{8} )</td>
<td>532626 ( 10^{4} )</td>
</tr>
<tr>
<td>99</td>
<td>4.95000 ( 10^{8} )</td>
<td>5.14199 ( 10^{8} )</td>
<td>4.94451 ( 10^{8} )</td>
<td>4.95000 ( 10^{8} )</td>
</tr>
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REFERENCES


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