ASYMPTOTICS OF STIRLING NUMBERS
OF THE SECOND KIND

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1. Introduction. Hsu [1] has given the asymptotic expansion

\[
S(N, K) \sim (1/3^2)^{N-K} \left[ 1 + \sum_{i=1}^{K-1} K^{i} f_{i}(N - K) + O(K^{-1}) \right] / (N - K)!
\]

for Stirling numbers \( S(N, K) \) of the second kind, where \( f_{i}(N - K) \) are polynomials of argument \( N - K \) and \( f_{i}(0) = 0 \). The expansion (1) is useful only for \( N - K \) small, as is indicated in §3. We obtain a complete asymptotic expansion of \( S(N, K) \) in powers of \( (N+1)^{-1} \), using the saddle point method previously employed by Moser and Wyman [2] and de Bruijn [3] for the asymptotic representation of the related Bell numbers. Convergence is demonstrated for \( K < (N+1)^{2/3}/[\pi+(N+1)^{-1/3}] \). The expansion when divergent is still useful when used as an asymptotic series.

2. Asymptotics of \( S(N, K) \). A generating function for \( S(N, K) \) is

\[
\left( \frac{e^{z} - 1}{z} \right)^{K} = \sum_{N=K}^{\infty} \frac{K!}{N!} S(N, K) z^{N-K}.
\]

Hence the Cauchy integral formula gives

\[
S(N, K) = \frac{N!}{2\pi i K!} \int_{C} (e^{z} - 1)^{K} z^{-N-1} \, dz
\]
where the contour $C$ encloses the origin. Equating the derivative of the integrand to zero gives the equation

$$ (t - z)e^{z-t} = te^{-t}, $$

where $t = (N+1)/K$, for the location of the saddle point of the modulus of the integrand. The principal saddle point $z = u$ is on the positive real axis with $t - 1 < u < t$. The quadratic approximation to $xe^{-x}$ at $x = 1$ shows that $u \approx 2/N$ for $K = N$ and large $N$. There are other subsidiary saddle points at complex roots of (4), which we neglect in comparison with the higher saddle point at $z = u$. Since there are no other roots of (4) for $|t - z| \leq t - u$, we may apply the Lagrange inversion formula to obtain

$$ u = t - \sum_{m=1}^{\infty} m^{m-1}(te^{-t})^{m}/m! $$

convergent for $t > 1$. Another form of (4) is the identity

$$ K = (N + 1)(1 - e^{-u})/u $$

needed later. Since the axis of the saddle point is perpendicular to the real axis, the part of the contour $C$ descending from $z = u$ is taken as the line $z = u + iy$, $|y| < \infty$, parallel to the imaginary axis. The modulus of the integrand in (3) is maximal at $z = u$ on this path, since both $(e^z - 1)^K$ and $z - N^{-1}$ have this property. The closed contour $C$ is completed by a half circle of infinite radius enclosing the origin. The contribution to the integral (3) on this semicircular path is zero since $N > 0$. The integral in (3) then becomes

$$ i(e^u - 1)^K u^{-N-1} \int_{-\infty}^{\infty} \exp \psi(u + iy) \; dy $$

where

$$ \psi(z) = K \ln\left[\frac{(e^{2u} - 1)(e^u - 1)}{(e^z - 1)}\right] - (N + 1)\ln|z|/u. $$

The contribution of the various parts of the $z = u + iy$ path to the integral must now be studied. As $|\exp \psi(z)| = \exp \Re \psi(z)$ we have to study

$$ \Re \psi(u + iy) = K \ln\left[\frac{(e^{2u} - 2e^u \cos y + 1)^{1/2}/(e^u - 1)}{(e^{2u} - 2e^u \cos y - 1)^{1/2}}\right] $$

$$ - (N + 1)\ln(1 + y^2u^{-2})^{1/2}. $$

We shall show that we can restrict ourselves essentially to the interval $|y| < \pi$. Since $1 + y^2u^{-2} \geq 1 + \pi(2y - \pi)u^{-2}$ for $y \geq \pi$ we have

$$ (e^u - 1)^K u^{-N-1} \left| \int_{\pi}^{\infty} \exp \psi(u + iy) \; dy \right| < \frac{u^{1-N}(e^u + 1)^K}{\pi(N - 1)(1 + \pi^2u^{-2})^{N/2-1/2}} $$
which is of $O(N^{-N}e^N)$ for $K$ small and of $O(2^N/\pi^N N)$ for $K$ large. Since $\Re \psi(u+iy)$ is even, the part of the integral (7) for $|y|>\pi$ tends toward zero as $N \to \infty$. We now direct our attention to the interval $|y|<\pi$ where the saddle point at $y=0$ gives the main contribution. The Taylor expansion of $\psi(u+iy)$, convergent for $|y|<u$, is

$$\psi = -\frac{N+1}{2u} \left( \frac{1}{u} - \frac{1}{e^u - 1} \right) y^2$$

(9)

$$+ (N+1) \sum_{j=1}^{\infty} \frac{(iy)^j}{(j+2)!} \left( \frac{d}{dz} \right)^{j+1} \left[ \frac{1-e^{-u}}{u(e^z-1)} - \frac{1}{z} \right]_{z=u}$$

where the identity (6) has been used. We now make the substitutions

$$w = [(N+1)/2]^{1/2} [1 - u/(e^u -1)]^{1/2} y/|u|$$

(10)

and

$$a_j = \frac{(iwu)^{j+2} (d/dz)^{j+1} [(1-e^{-z})/u(l^z-1) - 1/z]_{z=u}}{(j+2)! [\frac{1}{2} - \frac{1}{2} u/(e^u-1)]^{j+1}}$$

(11)

to obtain

$$S(N, K) = B \int_{-\infty}^{\infty} \exp\{ -w^2 + f[(N+1)^{-1/2}] \} \, dw$$

(12)

where

$$B = N! (e^u -1)^K / \pi (2(N+1))^{1/2} K! u^N (1 + u/(1-\exp u))^{1/2}$$

(13)

and $f$ is the analytic continuation of

$$f[(N+1)^{-1/2}] = \sum_{j=1}^{\infty} a_j (N+1)^{-j/2}.$$  

(14)

To find an upper bound to $|a_j|$ we note that $(e^z-1)^{-1} = \sum_{k=1}^{\infty} e^{-kz}$ for $\Re z>0$. Then

$$(d/du)^n (e^u -1)^{-1} = (-1)^n \sum_{z=0}^{\infty} g(x)$$

where $g(x)=x^n e^{-ux}$. On using the Euler-Maclaurin sum formula [5] we find

$$\sum_{x=0}^{\infty} g(x) = \frac{n!}{u^{n+1}} + R_n.$$ 

The remainder $R_n = \int_{s}^{\infty} (x-[x]-\frac{1}{2}) g'(x) \, dx$ may be evaluated by a Laplace transform [6] to be

$$R_n = (-1)^{n-1} [nF^{(n-1)}(u) + uF^{(n)}(u)]$$
where \( F(u) = u^2 - \frac{1}{2} u^{-1} \coth \frac{1}{u} \). We conclude that \(|R_n| \ll n! u^{n+1}\) for small \( u \) and tends to zero for large \( u \). Since \((1 - e^{-u})/u\) is less than unity we have

\[
|a_j| < \sigma^{j+2}/j
\]

where

\[
\sigma = w^{2^{1/2}}/(1 - u/(e^w - 1))^{1/2} = y(N + 1)^{1/2}/u.
\]

Remembering that we need not integrate (7) beyond \(|y| = \pi\) for large \( N\), we see by (15) and (16) that the series (14) is convergent for \( \pi/\mu < 1 \). We now expand \( \exp f((N+1)^{-1/2}) \) in a Taylor series of the form

\[
\exp f((N + 1)^{-1/2}) = \sum_{j=0}^{\infty} b_j(N + 1)^{-j/2}
\]

where \( b_0 = 1 \) and \( b_j \) are polynomials in \( w \) of the degree and parity of \( 3j \). By a lemma of Moser and Wyman \[4\]

\[
|b_j| \leq \sigma^{j+2}(1 + \sigma^2)^{-1}.
\]

Using (17) we may write (12) in the form

\[
S(N, K) = B\left[ \sum_{j=0}^{N-1} (N + 1)^{-j/2} \int_{-\infty}^{\infty} e^{-w^2} b_{2j} dw + R_s \right].
\]

The absolute value of the remainder \( R_s \) is found from (18) to be

\[
|R_s| \leq (N + 1)^{-d} \int_{-\infty}^{\infty} e^{-w^2} P_s(|w|) dw/M
\]

where \( P_s(|w|) \) is a polynomial in \(|w|\) and

\[
M = 1 - \sigma^2(1 + \sigma^2)^2/(N + 1).
\]

On limiting the integration in (7) to \(|y| < \pi\) we see that the remainder \( R_s \) exists if

\[
(\pi/\mu)[1 + (N + 1)(\pi/\mu)^2] < 1.
\]

Since \( u + 1 > (N + 1)/K \) convergence occurs for

\[
K < (N + 1)^{2/3}[(\pi + (N + 1)^{-1/2}]
\]

approximately. For these values of \( K \) we conclude that

\[
S(N, K) \sim B\left[ \sum_{j=0}^{N-1} (N + 1)^{-j/2} \int_{-\infty}^{\infty} e^{-w^2} b_{2j} dw + O((N + 1)^{-d}) \right].
\]
The first two terms of (20) have been calculated to be

\[ S(N, K) \sim \frac{N!(e^u - 1)^K}{(2\pi(N + 1))^{1/2}K!u^N(1 - G)^{1/2}} \]

\[ \cdot \left[ 1 - \frac{2 + 18G - 20G^2(e^u + 1)}{24(N + 1)(1 - G)^3} \right. \]

\[ + \frac{3G^3(e^u + 4e^u + 1) + 2G^4(e^u - e^u + 1)}{24(N+1)(1 - G)^3} \]

where by [5]

\[ G = \frac{u}{(e^u - 1)} = 1 - \frac{1}{2}u + \sum_{k=1}^{\infty} \frac{B_k u^{2k}}{(2k)!}. \]

The bracketed expression in (21), argumented by an additional inverse power of \( N + 1 \), is approximated by

\[ 1 - \frac{1}{6u(N + 1)} + \frac{1}{72u^2(N + 1)^2} \]

for small \( u \) and by

\[ 1 - \frac{1}{12(N + 1)} + \frac{1}{288(N + 1)^2} \]

for large \( u \). These are the leading terms of an alternating asymptotic series.

3. Numerical example. The 6-significant-figure Table 1 compares the exact values of \( S(100, K) \) with the values computed from (20) and (1) for several \( K \). The excellent results obtained from (20) for values of \( K \) outside the interval of convergence show that the expansion gives useful results when used as an asymptotic series.

<table>
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<tr>
<th>( K )</th>
<th>( S(100, K) ) Exact</th>
<th>( S(100, K) ) 1 term of (20)</th>
<th>( S(100, K) ) 2 terms of (20)</th>
<th>( S(100, K) ) 4 terms of (1)</th>
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<tr>
<td>2</td>
<td>6.33825 ( 10^{28} )</td>
<td>6.34348 ( 10^{28} )</td>
<td>6.33825 ( 10^{28} )</td>
<td>1.81186 ( 10^{-111} )</td>
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<tr>
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<td>2.58496 ( 10^{14} )</td>
<td>2.58321 ( 10^{14} )</td>
<td>2.94696 ( 10^{93} )</td>
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<tr>
<td>50</td>
<td>4.30983 ( 10^{10} )</td>
<td>4.30990 ( 10^{10} )</td>
<td>4.30977 ( 10^{10} )</td>
<td>1.51529 ( 10^{44} )</td>
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<td>4.94451 ( 10^{4} )</td>
<td>4.95000 ( 10^{4} )</td>
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References


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