ASYMPTOTICS OF STIRLING NUMBERS OF THE SECOND KIND

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ABSTRACT. A complete asymptotic development of the Stirling numbers $S(N, K)$ of the second kind is obtained by the saddle point method previously employed by Moser and Wyman [Trans, Roy. Soc. Canad., 49 (1955), 49–54] and de Bruijn [Asymptotic methods in analysis, North-Holland, Amsterdam, 1958, pp. 102–109] for the asymptotic representation of the related Bell numbers.

1. Introduction. Hsu [1] has given the asymptotic expansion

$$S(N, K) \sim \left(\frac{k}{N-K}\right)^{N-K} \left[1 + \sum_{i=1}^{k} \frac{K^{-i} f_i(N-K) + O(K^{-i-1})}{(N-K)!}\right]$$

for Stirling numbers $S(N, K)$ of the second kind, where $f_i(N-K)$ are polynomials of argument $N-K$ and $f_i(0)=0$. The expansion (1) is useful only for $N-K$ small, as indicated in §3. We obtain a complete asymptotic expansion of $S(N, K)$ in powers of $(N+1)^{-1}$, using the saddle point method previously employed by Moser and Wyman [2] and de Bruijn [3] for the asymptotic representation of the related Bell numbers. Convergence is demonstrated for $K < (N+1)^{2/3} / [(N+1)^{1/2}]$. The expansion when divergent is still useful when used as an asymptotic series.

2. Asymptotics of $S(N, K)$. A generating function for $S(N, K)$ is

$$\left(\frac{e^z - 1}{z}\right)^K = \sum_{N=K}^{\infty} \frac{K!}{N!} S(N, K) z^{N-K}.$$ 

Hence the Cauchy integral formula gives

$$S(N, K) = \frac{N!}{2\pi i K} \int_C (e^z - 1)^K z^{-N-1} dz$$
where the contour $C$ encloses the origin. Equating the derivative of the integrand to zero gives the equation

$$ (t - z)e^{z - t} = te^{-t}, $$

where $t = (N+1)/K$, for the location of the saddle point of the modulus of the integrand. The principal saddle point $z = u$ is on the positive real axis with $t - 1 < u < t$. The quadratic approximation to $xe^{-x}$ at $x = 1$ shows that $u \approx 2/N$ for $K = N$ and large $N$. There are other subsidiary saddle points at complex roots of (4), which we neglect in comparison with the higher saddle point at $z = u$. Since there are no other roots of (4) for $|t - z| \leq t - u$, we may apply the Lagrange inversion formula to obtain

$$ u = t - \sum_{m=1}^{\infty} m^{-1}(te^{-t})^m / m! $$

convergent for $t > 1$. Another form of (4) is the identity

$$ K = (N + 1)(1 - e^{-u})/u $$

needed later. Since the axis of the saddle point is perpendicular to the real axis, the part of the contour $C$ descending from $z = u$ is taken as the line $z = u + iy$, $|y| < \infty$, parallel to the imaginary axis. The modulus of the integrand in (3) is maximal at $z = u$ on this path, since both $(e^z - 1)^K$ and $z - u - i$ have this property. The closed contour $C$ is completed by a half circle of infinite radius enclosing the origin. The contribution to the integral (3) on this semicircular path is zero since $N > 0$. The integral in (3) then becomes

$$ i(e^u - 1)^K u^{-N - 1} \int_{-\infty}^{\infty} \exp \psi(u + iy) \, dy $$

where

$$ \psi(z) = K \ln[(e^z - 1)/(e^u - 1)] - [(N + 1)\ln(z/u)]. $$

The contribution of the various parts of the $z = u + iy$ path to the integral must now be studied. As $|\exp \psi(z)| = \exp \Re \psi(z)$ we have to study

$$ \Re \psi(u + iy) = K \ln[(e^{2u} - 2e^u \cos y + 1)^{1/2}/(e^u - 1)] - (N + 1)\ln(1 + y^2u^{-2})^{1/2}. $$

We shall show that we can restrict ourselves essentially to the interval $|y| < \pi$. Since $1 + y^2u^{-2} \geq 1 + \pi(y - \pi)u^{-2}$ for $y \geq \pi$ we have

$$ (e^u - 1)^K u^{-N - 1} \left| \int_{\pi}^{\infty} \exp \psi(u + iy) \, dy \right| < \frac{u^{1-N}(e^u + 1)^K}{\pi(N - 1)(1 + \pi^2u^{-2})^{N/2 - 1/2}} $$
which is of $O(N^{1-N})$ for $K$ small and of $O(2^{N+1} / \pi N N)$ for $K$ large. Since Re $\psi(u+iy)$ is even, the part of the integral (7) for $|y| > \pi$ tends toward zero as $N \to \infty$. We now direct our attention to the interval $|y| < \pi$ where the saddle point at $y=0$ gives the main contribution. The Taylor expansion of $\psi(u+iy)$, convergent for $|y| < u$, is

$$
\psi = -\frac{N + 1}{2u} \left( \frac{1}{u} - \frac{1}{e^u - 1} \right) y^2
$$

where the identity (6) has been used. We now make the substitutions

$$
w = \left[ (N + 1)/2 \right]^{1/2} \left[ 1 - u/(e^u - 1) \right]^{1/2} y/|u|
$$

and

$$
a_j = \left[ \frac{(iu)^{j+2} (d/dz)^{j+1} [(1 - e^{-u})/u(l^2 - 1) - 1/z]_{z=0}}{(j + 2)!} \right]_{z=0}^{1/2} y/|u|
$$

to obtain

$$
S(N, K) = B \int_{-\infty}^{\infty} \exp \left\{ -w^2 + f[(N + 1)^{-1/2}] \right\} dw
$$

where

$$
B = N! (e^u - 1)^K / \pi (2(N + 1))^{1/2} K! u^N (1 + u/(1 - \exp u))^{1/2}
$$

and $f$ is the analytic continuation of

$$
f[(N + 1)^{-1/2}] = \sum_{j=1}^{\infty} a_j (N + 1)^{-j/2}.
$$

To find an upper bound to $|a_j|$ we note that $(e^z - 1)^{-1} = \sum_{k=1}^{\infty} e^{-kz}$ for Re $z > 0$. Then

$$
(d/du)^n (e^u - 1)^{-1} = (-1)^n \sum_{z=0}^{\infty} g(x)
$$

where $g(x) = x^n e^{-ux}$. On using the Euler-Maclaurin sum formula [5] we find

$$
\sum_{x=0}^{\infty} g(x) = \frac{n!}{u^{n+1}} + R_n.
$$

The remainder $R_n = \int_0^\infty (x-[x] - \frac{1}{2}) g(x) \, dx$ may be evaluated by a Laplace transform [6] to be

$$
R_n = (-1)^{n-1} [nF^{(n-1)}(u) + uF^{(n)}(u)]
$$
where \( F(u) = u^2 - \frac{1}{2}u^{-1} \coth \frac{1}{2}u \). We conclude that \( |R_n| \ll n!/u^{n+1} \) for small \( u \) and tends to zero for large \( u \). Since \( (1 - e^{-u})/u \) is less than unity we have

\[
|a_j| < \sigma^{j+2}/j
\]

where

\[
\sigma = w^{21/2}/(1 - u/(e^u - 1))^{1/2} = y(N + 1)^{1/2}/u.
\]

Remembering that we need not integrate (7) beyond \( |y| = \pi \) for large \( N \), we see by (15) and (16) that the series (14) is convergent for \( \pi/u < 1 \). We now expand \( \exp\left(\frac{N+1)^{-1/2}}{2} \right) \) in a Taylor series of the form

\[
\exp\left(\frac{N+1)^{-1/2}}{2} \right) = \sum_{j=0}^{\infty} b_j (N + 1)^{-j/2}
\]

where \( b_0 = 1 \) and \( b_j \) are polynomials in \( w \) of the degree and parity of \( 3j \).

By a lemma of Moser and Wyman [4]

\[
|b_j| \leq \sigma^{j+2}(1 + \sigma^2)^{j-1}.
\]

Using (17) we may write (12) in the form

\[
S(N, K) = B\left( \sum_{j=0}^{N-1} (N + 1)^{-j} \int_{-\infty}^{\infty} e^{-w^2} b_{2j} \, dw + R_s \right).
\]

The absolute value of the remainder \( R_s \) is found from (18) to be

\[
|R_s| \leq (N + 1)^{-5} \int_{-\infty}^{\infty} e^{-w^2} P_s(|w|) \, dw/M
\]

where \( P_s(|w|) \) is a polynomial in \( |w| \) and

\[
M = 1 - \sigma^2(1 + \sigma^2)^{2}/(N + 1).
\]

On limiting the integration in (7) to \( |y| < \pi \) we see that the remainder \( R_s \) exists if

\[(\pi/u)[1 + (N + 1)(\pi/u)^2] < 1.\]

Since \( u+1 > (N+1)/K \) convergence occurs for

\[K < (N + 1)^{2/3}/[\pi + (N + 1)^{-1/2}]\]

approximately. For these values of \( K \) we conclude that

\[
S(N, K) \sim B\left( \sum_{j=0}^{N-1} (N + 1)^{-j} \int_{-\infty}^{\infty} e^{-w^2} b_{2j} \, dw + O((N + 1)^{-j}) \right).
\]

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The first two terms of (20) have been calculated to be

\[ S(N, K) \sim \frac{N!(e^u - 1)^K}{(2\pi(N + 1))^{3/2}K!u^N(1 - G)^{1/2}} \times \left[ 1 + \frac{2 + 18G - 20G^2(e^u + 1)}{24(N + 1)(1 - G)^3} \right. \]

\[ \left. + \frac{3G^3(e^{2u} + 4e^u + 1) + 2G^4(e^{2u} - e^u + 1)}{24(N+1)(1 - G)^3} \right] \]

where by [5]

\[ G = \frac{u}{(e^u - 1)} = 1 - \frac{1}{2}u + \sum_{k=1}^{\infty} \frac{B_{2k}u^{2k}}{(2k)!}. \]

The bracketed expression in (21), argumented by an additional inverse power of \(N+1\), is approximated by

\[ 1 - \frac{1}{6u(N + 1)} + \frac{1}{72u^2(N + 1)^2} \]

for small \(u\) and by

\[ 1 - \frac{1}{12(N + 1)} + \frac{1}{288(N + 1)^2} \]

for large \(u\). These are the leading terms of an alternating asymptotic series.

3. Numerical example. The 6-significant-figure Table 1 compares the exact values of \(S(100, K)\) with the values computed from (20) and (1) for several \(K\). The excellent results obtained from (20) for values of \(K\) outside the interval of convergence show that the expansion gives useful results when used as an asymptotic series.

<table>
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<tr>
<th>(K)</th>
<th>(S(100, K)) Exact</th>
<th>(S(100, K)) 1 term of (20)</th>
<th>(S(100, K)) 2 terms of (20)</th>
<th>(S(100, K)) 4 terms of (1)</th>
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<td>6.33825 (10^{29})</td>
<td>6.34348 (10^{29})</td>
<td>6.33825 (10^{29})</td>
<td>1.81186 (10^{-11})</td>
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<td>2.94959 (10^{9})</td>
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<td>4.94451 (10^{3})</td>
<td>4.95000 (10^{3})</td>
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References


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