A NOTE ON A LEMMA OF ZARISKI AND HIGHER DERIVATIONS

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Abstract. A sufficient condition is given for an $a$-adic complete ring $R$ to be a power series ring over a subring.

1. Introduction. We prove in this note the following theorem: Let $R$ be a ring and let $a$ be an ideal in $R$ such that $\bigcap_{i=0}^{\infty} a^i = 0$ and $R$ is complete with respect to the $a$-adic topology. Assume that there exists a higher derivation $D = \{D_i\}_{i=0}^{\infty}$ of $R$ such that $D_1(x) = 1$ for some $x \in a$. Let $E = D_0 - xD_1 + \cdots + (-1)^n x^n D_n + \cdots$. If $E(x) = 0$, then there exists a subring $R_1$ of $R$ such that $a = a[[x]]$, and $x$ is analytically independent over $R_1$.

This result generalizes Zariski's original lemma [5, Lemma 4, p. 526], and [1, Theorem 6, p. 412], a version of Zariski's lemma when $R$ is of positive characteristic, and also removes the condition that $R$ is an integral domain as we mentioned at the end of [1, p. 414]. [5, Lemma 4, p. 256] played a very important role in the study of analytic product of an affine algebraic variety $V$ along a given subvariety $W$ of $V$ in A. Seidenberg's paper on differential ideals [4].

In the last section, we generalize a lemma of M. Miyanishi [2, p. 194] slightly, and give some remarks on his proofs of his lemma and Proposition 1.3 [2, p. 194].

2. Preliminaries. Throughout this note, all rings are commutative with identity. A derivation $D$ of a ring $R$ is an additive group homomorphism from $R$ to $R$ such that $D(ab) = aDb + bDa$ for all $a$ and $b$ in $R$. A higher derivation $D = \{D_i\}_{i=0}^{\infty}$ of a ring $R$ is a sequence of additive group homomorphisms from $R$ to $R$ such that

(1) $D_0$ = identity map on $R$, and

(2) $D_n(a \cdot b) = \sum_{i=0}^{n} D_i(a) \cdot D_{n-i}(b)$ for all $n=1, 2, \cdots$, and for all $a$ and $b$ in $R$. (Note $D_1$ is always a derivation of $R$.) Leibniz formula.

Let $R$ be a ring and let $a$ be an ideal in $R$. Then $R$ has a topological structure with $\{a^i\}_{i=0}^{\infty}$ as the fundamental system of neighborhoods of the
zero in $R$. This is the so called $a$-adic topology in $R$. Both addition and multiplication are continuous in $a$-adic topology. $R$ is a Hausdorff space if and only if $\bigcap_{i=0}^{\infty} a^i = 0$. A mapping $f: R \to R$ is continuous if there exists a subsequence $\{a^n\}_{n=0}^{\infty}$ of the sequence $\{a^n\}_{n=0}^{\infty}$ such that $f(a^n) = a^n$, for each natural number $n$.

**Lemma 1.** Let $R$ be a ring. Let $D = \{D_i\}_{i=0}^{\infty}$ be a higher derivation of $R$. Then

1. For $a_1, \cdots, a_n \in R$, and for each natural number $m$,

   $$D_m(a_1 \cdots a_n) = \sum_{i=1}^{n} a_i \cdots \hat{a}_i \cdots a_n D_m(a_i)$$

   $$+ \sum_{i=1}^{m-1} D_i(a_1 \cdots a_{n-1}) \cdot D_{m-i}(a_n)$$

   $$+ \sum_{i=1}^{m-1} \sum_{j=2}^{n-1} a_{i+1} \cdots a_n D_i(a_1 \cdots a_{j-1}) D_{m-i}(a_j),$$

   where $a_1 \cdots \hat{a}_i \cdots a_n = a_1 \cdots a_{i-1} a_{i+1} \cdots a_n$.

2. For each ideal $a$ in $R$, and for each natural number $m$, $D_m(a^n) \subseteq a^{n-m}$ for each $a^n$, i.e. $D_m$ is continuous with respect to the $a$-adic topology.

3. For each ideal $a$ in $R$ and for each $x \in a$, let $E_x = \sum_{i=0}^{\infty} (-x)^i D_i$. Then the sequence $\{E_x\}_{x=0}^{\infty}$ is uniformly convergent, and $E = \sum_{i=0}^{\infty} (-1)^i x^i D_i = \lim_{i \to \infty} E_i$ is a continuous additive group homomorphism of $R$ with respect to the $a$-adic topology.

**Proof.** Straightforward.

**Lemma 2.** Let $R$ be a ring. Let $a$ be an ideal of $R$ such that $\bigcap_{i=0}^{\infty} a^i = 0$. Assume $R$ is complete with respect to the $a$-adic topology. Let $x \in a$, and let $E_i = \sum_{i=0}^{\infty} (-1)^i x^i D_i$. Then the sequence $\{E_i\}_{i=0}^{\infty}$ is uniformly convergent, and $E = \sum_{i=0}^{\infty} (-1)^i x^i D_i = \lim_{i \to \infty} E_i$ is a continuous endomorphism of $R$.

**Proof.** The sequence $\{E_i(a)\}$ is a Cauchy sequence for each $a \in R$. In fact $E_i(a) - E_j(a) \in a^n$ for $i, j > n$; i.e., for each given $a^n$, there exists a natural number $N (= n)$ such that $E_i(a) - E_j(a) \in a^n$ for $i, j > N$. Hence $\{E_i(a)\}$ converges in $R$. Since the natural number $N$ above is independent of $a \in R$, therefore $\{E_i\}_{i=0}^{\infty}$ converges uniformly in $R$. Hence $E = \lim_{i \to \infty} E_i$ is continuous. In fact for each $a^n$ and for each $a \in a^n$, there exists a natural number $N = n$ which is independent of $a$ such that $E(a) = (E(a) - E_N(a)) + E_N(a) \in a^n$. Thus $E(a^n) \subseteq a^n$.

Since $E(a+b) = \lim_{i \to \infty} E_i(a+b) = \lim_{i \to \infty} E_i(a) + \lim_{i \to \infty} E_i(b) = E(a) + E(b)$, therefore $E(a+b) = E(a) + E(b)$. Also for each natural number $l$,
Theorem 1. Let $R$ be a ring and let $I$ be a proper ideal in $R$ such that $\bigcap_{i=0}^{\infty} I^i=0$, and $R$ is complete with respect to the $I$-adic topology. Assume that there exists a higher derivation $D=\{D_i\}_{i=0}^{\infty}$ of $R$ such that $D_1(x)=1$ for some $x \in I$. Let $E=D_0-xD_1+\cdots+(-1)^nx^nD_n+\cdots$. If $E(x)=0$, then there exists a subring $R_1$ of $R$ such that $R=R_1[[x]]$, and $x$ is analytically independent over $R_1$; i.e. if $\sum_{i=0}^{\infty} a_ix^i=0$ where $a_i \in R_1$ then $a_i=0$ for all $i=1, 2, \cdots$.

Proof. $E(x)=0$ implies $E^{-1}(0)=Rx$. Indeed, if $a \in E^{-1}(0)$, then $E(a)=0$; i.e. $a-xD_1(a)+\cdots+(-1)^nx^nD_n(a)+\cdots=0$. Thus $a=xD_1(a)+\cdots+(-1)^nx^nD_n(a)+\cdots$ is in $Rx$. So $E^{-1}(0) \subseteq Rx$. The other inclusion is obvious. Next, we observe that $E^2=E$. In fact for each $a \in R$, 

$$E(a) = a - xD_1a + \cdots$$

and 

$$E^2(a) = E(a) - (xD_1a + \cdots)$$

$$= E(a) - E(x) \cdot E(D_1a - xD_2a + \cdots) = E(a),$$

so $E^2(a)=E(a)$ for all $a \in R$. Let $R_1=E(R)$ then $E$ is an identity map on $R_1$. Let $a$ be an arbitrary element in $R$. $E(a)=a-xD_1a+\cdots+(-1)^nx^nD_n(a)+\cdots$ implies that $a=E(a)+a_1x$ for some $a_1 \in R$. Thus $a=E(a)+xE(a_1)+a_2x^2$ for some $a_2 \in R$, and so on. Therefore we have $a=E(a)+xE(a_1)+\cdots+x^nE(a_n)+\cdots$ for some $a_1, a_2, \cdots, a_n, \cdots$ in $R$, and $a \in R_1[[x]]$. Hence $R=R_1[[x]]$. Finally we suppose $b_0+b_1x+\cdots+b_mx^m+\cdots=0$ where $b_0, b_1, \cdots, b_n, \cdots$ are in $R_1$. We prove inductively $b_0=b_1=\cdots=b_n=\cdots=0$. Since $E$ is identity on $R_1$, $E(b_0+b_1x+\cdots+b_mx^m+\cdots)=0$ implies $b_0=E(b_0)=0$. Assume $b_0=b_1=\cdots=b_i=0$. We have $b_{i+1}x^{i+1}+b_{i+2}x^{i+2}+\cdots=0$. By Lemma 1 and $D_1(x)=1$, we have $D_nx^n\equiv 0 \mod(x)$ and $D_nx^{n+j}\equiv 0 \mod(x)$ for all natural numbers $n$ and $j$. Thus 

$$0 = D_{i+1}(b_{i+1}x^{i+1} + b_{i+2}x^{i+2} + \cdots)$$

$$= D_{i+1}(b_{i+1}x^{i+1}) + D_{i+1}(b_{i+2}x^{i+2} + \cdots)$$

$$\equiv b_{i+1}D_{i+1}(x) \mod(x) \equiv b_{i+1} \mod(x).$$

Therefore $b_{i+1}+c_{i+1}x=0$ for some $c_{i+1} \in R$. So $0=E(b_{i+1}+c_{i+1}x)=b_{i+1}$. Therefore $x$ is analytically independent over $R_1$. 

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If $R$ contains the field of rational numbers as a subring, then every derivation $D$ of $R$ gives rise to a higher derivation of $R$, namely, \( \{D_0, D, D^2/2!, \ldots, D^n/n!\} \) where $D^n$ is the $n$th successive derivation of $D$, and $D_0$ is the identity map in $R$. Let $a$ be an ideal of $R$ such that $R$ is a complete Hausdorff space with respect to the $a$-adic topology. Assume $Dx = 1$ for some $x \in a$. Then the endomorphism $E = e^{-zD} = \sum \frac{(-1)^n x^n D^n}{n!}$ always maps $x$ to zero. Let $R_1 = E(R)$, then $D(R_1) = 0$. Thus we have the following

**Corollary 1.** Let $R$ be a ring containing the field of rational numbers as a subring. Let $a$ be an ideal in $R$ such that $R$ is a complete Hausdorff space with respect to the $a$-adic topology. Assume there exists a derivation $D$ of $R$ such that $Dx = 1$ for some $x \in a$. Then there exists a subring $R_1$ of $R$ such that (1) $D$ is zero on $R_1$ and (2) $R = R_1[[x]]$ and $x$ is analytically independent over $R_1$.

In the following, a semilocal (local) ring $\mathfrak{D}$ is a Noetherian ring with finitely many (unique) maximal ideals. Let $m$ be the intersection of the maximal ideals of $\mathfrak{D}$. It is well known that $\bigcap_{i=0}^{\infty} m^i = 0$. In this case we use $m$-adic topology for $R$. As a corollary to Corollary 1, we have the original lemma of Zariski.

**Corollary 2.** Let $(\mathfrak{D}, m)$ be a complete semilocal ring of characteristic zero. Let $D$ be a derivation of $\mathfrak{D}$. Assume that there exists an element $x$ in $m$ of $\mathfrak{D}$ such that $Dx$ is a unit in $\mathfrak{D}$. Then $\mathfrak{D}$ contains a ring $\mathfrak{D}_1$ of representatives of the (complete) semilocal ring $\mathfrak{D}/\mathfrak{D}x$ having the following properties: (a) $D$ is zero on $\mathfrak{D}_1$; (b) $x$ is analytically independent on $\mathfrak{D}_1$; (c) $\mathfrak{D} = \mathfrak{D}_1[[x]]$.

**Proof.** Replace $D$ by $(1/Dx)D$ and apply Corollary 1.

We also get [1, Theorem 6, p. 412] as a corollary to the theorem,

**Corollary 3.** Let $(\mathfrak{D}, m)$ be a complete local ring. Let $x \in m$ and let $D = \{D_i\}_{i=0}^{\infty}$ be a higher derivation of $\mathfrak{D}$ such that $D_1x$ is a unit in $\mathfrak{D}$, and $D_i x = 0$ for $i > 1$. Then there exists a subring $\mathfrak{D}_1$ of $\mathfrak{D}$ such that: (a) $\mathfrak{D}_1$ is a complete local ring, (b) $x$ is analytically independent over $\mathfrak{D}_1$, and (c) $\mathfrak{D} = \mathfrak{D}_1[[x]]$.

**Proof.** Let $D_1 x = e^{-1}$, where $e$ is a unit in $\mathfrak{D}$. Replacing $\{D_i\}_{i=0}^{\infty}$ by $\{e^i D_i\}_{i=0}^{\infty}$, we may assume $D_1x = 1$. Since $D_i x = 0$ for $i > 1$, therefore $Ex = x - xD_1x = 0$ where $E = D_0 - xD_1 + \cdots + (-1)^n x^n D_n + \cdots$. Thus the theorem is applicable.

**Remarks.** (1) Theorem 1 and Corollary 1 hold under the assumption that $D_1 x$ is a unit. The proofs are easily modified.
(2) If $R$ has $a$ as its sole maximal ideal and is a complete Hausdorff space with respect to the $a$-adic topology, then $E(x) = 0$ and $x \neq 0$ implies that $D_i(x)$ is a unit.

(3) If $R$ is a complete Hausdorff integral domain with respect to the $a$-adic topology, then $E(x) = 0$ and $x \neq 0$ implies that $D_i(x)$ is a unit.

4. Though the following theorem could be easily proved by a similar technique used in the proof of Theorem 1, we would like to prove it as a corollary to Corollary 1.

**Theorem 2.** Let $R$ be a ring containing the field of rational numbers as a subring. Let $a$ be an ideal in $R$ such that $\bigcap_{i=0}^{\infty} a^i = 0$. Assume that there exists a derivation $D$ of $R$ such that (1) for each $\gamma \in R$, $D(i\gamma) = 0$ for sufficiently large $i$, and (2) $D(x) = 1$ for some $x \in a$. Then (a) there exists a subring $R_1$ of $R$ such that $R = R_1[x]$ and $x$ is algebraically independent over $R_1$; (b) $D$ is trivial on $R_1$.

**Proof.** Let $\hat{R}$ be the completion of $R$ with respect to the $a$-adic topology. Then $\hat{R}$ is a complete Hausdorff space with respect to the topology defined by the filtration $\hat{R} = \bigcap_{i=0}^{\infty} (a^i)$. Where $(a^i)$ is the closure of $a^i$ in $\hat{R}$. Thus $\bigcap_{i=0}^{\infty} (a^i)^{\infty} = 0$. Note that $(a^i)^{\infty} = a^i \hat{R}$ in general and equality holds if $a$ is finitely generated. Let $\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots$ be a Cauchy sequence in $R$; Lemma 1 implies that $(\gamma_i)$ is also a Cauchy sequence for each $i$. Define $\hat{D}(\gamma) = \lim_{i \to \infty} D(\gamma_i)$. Then it is easy to check that $\{\hat{D}_0, \hat{D}_1, \ldots, \hat{D}/n!, \ldots\}$ forms a higher derivation in $\hat{R}$. Moreover $\hat{D}(a^i) = (a^{i-j})^\infty$. Indeed let $\gamma \in (a^i)^{\infty}$ and let $\gamma_1, \ldots, \gamma_n, \ldots$ be a sequence in $a^i$ such that $\lim_{i \to \infty} \gamma_i = \gamma$. Then $\hat{D}(\gamma) = \lim_{i \to \infty} D(\gamma_i)$. Since $D(\gamma_i) \in (a^{i-j})$ and $D(\gamma) \in (a^{i-j})^{\infty}$, Lemma 1 and Lemma 2 are easily verified. Since the kernel of the natural ring homomorphism from $R$ to $\hat{R}$ is $\bigcap_{i=0}^{\infty} a^i = 0$, $R$ is viewed as a subring of $\hat{R}$. $\hat{D}$ restricted to $R$ is $D$ so $\hat{D}x = 1$ and

$$\hat{E}(x) = x - (x/1!) \hat{D}(x) + \cdots + (-1)^n(x^n/n!) \hat{D}^n(x) + \cdots$$

$$= x - (x/1!) D(x) + (x^2/2!) D^2(x) + \cdots + (-1)^n(x^n/n!) D^n(x) + \cdots$$

$$= 0.$$ 

Thus it follows from Theorem 1 that $\hat{R} = \hat{R}_1[[x]]$, where $\hat{R}_1$ is a subring of $\hat{R}$ and $x$ is analytically independent over $\hat{R}_1$. Let $R_1 = R \cap \hat{R}_1$. Then $R = R_1[x]$. Indeed, let $\gamma \in R$. Then $\gamma = a_0 + a_1 x + \cdots + a_n x^n + \cdots$ where $a_i \in \hat{R}_1$ for all $a_i$. Since there exists a natural number $N$ such that $D^N(\gamma) = 0$. Therefore

$$0 = D^N(\gamma) = N! a_N + \frac{1}{2}(N + 1)! a_{N+1} x + \cdots.$$ 

Hence $a_i = 0$ for all $i \geq N$, and $\gamma = a_0 + a_1 x + \cdots + a_{N-1} x^{N-1}$. It follows that $R \subset \hat{R}_1[x]$. Applying $D^{N-1}$ to $\gamma$, we have $D^{N-1}(\gamma) = (N-1)! a_{N-1}$. 

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Therefore $a_{N-1} \in R$. Applying $D^{N-2}$ to $(y-a_{N-1}x^{N-1})=a_0+a_1x+\cdots+a_{N-2}x^{N-2}$, we get $a_{N-2} \in R$ and so on. Consequently, $a_0, \cdots, a_{N-1}$ are all in $R_1 \cap R=R_1$. So $R=R_1[x]$, and $x$ is of course algebraically independent over $R_1$. (b) follows from Corollary 1.\

We would like to thank Professor M. Miyanishi for communicating to us the following result which we also observed independently.

**Proposition 1.** Let $R$ be an integral domain of characteristic 0. Assume there is a derivation of $R$ such that $D^i(a)=0$ for each $a \in R$ and for sufficiently large $i$. Then $Dx=0$ for all units $x$ in $R$.

**Proof.** Let $x$ be a unit in $R$, and let $y \in R$ be such that $xy=1$. Then $xDy+yDx=0$. Suppose $Dx \neq 0$. Thus $Dy \neq 0$. Let $i$ be the natural number such that $D^i x=0$ and $D^m x \neq 0$ for $m<i$, also let $j$ be the natural number such that $D^j y=0$ and $D^p y \neq 0$ for $p<j$. By Leibniz’s formula,

$$0 = D^n(xy) = \sum_{k=0}^{n} \binom{n}{k} D^k(x)D^{n-k}(y).$$

Taking $n=i+j-2$ and assuming $i \leq j$ we get $D^{i-1}(x) \cdot D^{i-1}(y)=0$. Hence either $D^{i-1}(x)=0$ or $D^{i-1}(y)=0$, a contradiction.

Proposition 1 completes the proof of [2, Lemma 1.4, p. 194]. One could not use [2, Proposition 1.4, p. 194] to yield a proof to the last part of [2, Proposition 1.3, p. 193]. But Proposition 1 corrects that part of the proof.


**Theorem 3.** Let $R$ be an integral domain of characteristic 0 with a unique maximal ideal $m$ such that $\bigcap_{i=0}^{\infty} m^i=0$, i.e. $(R, m)$ is a local domain which may not be Noetherian. If there is a derivation $D$ of $R$ such that $D^i(a)=0$ for each $a \in R$ and for sufficiently large $i$, then $m$ is differential, i.e. $D(m) \subseteq m$, and $D$ induced by $D$ on $R/m$ is trivial.

**Proof.** Suppose $Dm \subsetneq m$. Then there is $x \in m$ such that $Dx=u^{-1}$, where $u$ is a unit in $R$. Then $uDx=1$. Replacing $D$ by $uD$, we have $(uD)^i(a)=u^iD^i(a)$ by Proposition 1. Thus $(uD)^i(a)=0$ for sufficiently large $i$. It follows from Theorem 2 that $R=R_1[x]$, a contradiction. The last part follows from Proposition 1.

Observing the fact that in a polynomial ring $A[x]$ the units in $A[X]$ are of the form $a_0+a_1x+\cdots+a_nx^n$ such that $a_0$ is a unit in $A$ and $a_2, \cdots, a_n$ are nilpotent in $A$, we give two examples countering Proposition 1 when $R$ is not an integral domain.
Example 1. Let \( R = \mathbb{Z}/(4)[X] \), where \( \mathbb{Z} \) is the domain of integers and \( X \) is an indeterminate over \( \mathbb{Z}/(4) \). Let \( D \) be a derivation of \( R \) such that \( DX = 1 + 2X + 2X^2 \). \((DX)^2 = 1, D^2 X = \neq 0, D^i(a) = 0 \) for each \( a \in R \) and for large integers \( i \).

Example 2. Let \( R = (\mathbb{Q}[t])[X] \), where \( \mathbb{Q} \) is the field of rational numbers, \( t^2 = 0 \) and \( X \) is an indeterminate over \( \mathbb{Q}[t] \). Let \( D \) be a derivation of \( R \) such that \( DX = 1 + tX \). Then \( DX \) is a unit \( ((DX) \cdot (1 - tX) = 1) \). \( D^2 X = t \neq 0 \), and \( Dt = 0, D^i(a) = 0 \) for each \( a \in R \) and for large \( i \).

In the setting of Theorem 2, when \( R \) is an integral domain, if there is a derivation \( D \) such that \( DX \) is a unit \( u \) for some \( x \in \mathfrak{a} \), then there is a \( y \in \mathfrak{a} \) (\( y = x/u \)) such that \( Dy = 1 \). What can one say in a more general case? Both Example 1 and Example 2 give negative answers. Using an idea of Professor M. Rosenlicht [3, Theorem 1, p. 721] we prove the following theorem.

**Theorem 4.** Let \( R \) be a ring, which contains the field of rational numbers, with an ideal \( \mathfrak{a} \) such that \( \bigcap_{i=0}^{\infty} \mathfrak{a}^i = 0 \) and \( R \) is complete with respect to the \( \mathfrak{a} \)-adic topology. Assume there is a derivation \( D \) of \( R \) such that \( Dx \) is a unit for some \( x \in \mathfrak{a} \). Then there exists an element \( y \in \mathfrak{a} \) such that \( Dy = 1 \).

**Proof.** If \( Dx \) is a unit then \( D(x/Dx) - 1 \in Rx \subset \mathfrak{a} \). If we can construct a Cauchy sequence \( \{x_0 = x/Dx, x_1, \cdots, x_n, \cdots\} \) such that \( x_i \in xR \) and \( Dx_i - 1 \in Rx^{i+1} \), then putting \( y = \lim_{i \to \infty} x_i \), and since \( \mathfrak{a} \) is also closed, we have \( y \in \mathfrak{a} \) and \( Dy = \lim_{i \to \infty} Dx_i = 1 \). The proposed construction goes inductively as follows: Since \( D(Rx^{i+1}) \subset Rx^i \), \( D \) induces a surjective \( R \)-homomorphism \( D(x^{i+1}) : Rx^i \to Rx/ Rx^{i+1} \) such that

\[ D(x^{i+1}) = (i + 1)x^i + Rx^{i+1} \]

Therefore there exists \( z_i \in Rx^{i+1} \) such that

\[ Dz_i = Dx_{i-1} - 1 \mod(Rx^{i+1}) \]

for \( i = 1, 2, \cdots \). Thus \( D(x_{i-1} - z_i) - 1 \in Rx^{i+1} \). Putting \( x_i = x_{i-1} - z_i \), we have a sequence \( \{x_0 = x/Dx, x_1, x_2, \cdots\} \) such that \( Dx_i - 1 \in Rx^{i+1} \subset \mathfrak{a}^{i+1} \) for \( i = 0, 1, 2, \cdots \). For a given \( \mathfrak{a}^n \), there exists a positive integer \( N \) \((=n)\) such that for \( i, j > N \), \( x_i - x_j \in \mathfrak{a}^n \). Therefore \( \{x_0, x_1, \cdots\} \) is a Cauchy sequence as desired.

**Added in Proof.** The author recently discovered that Theorem 2 can be derived from Taylor's lemma, see Y. Nouaze and P. Gabriel's *Idéaux premiers de l'algèbre enveloppante d'une algèbre de Lie nilpotente*, J. Algebra 6 (1967), 77–99.
REFERENCES


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