THE kTH CONJUGATE POINT FUNCTION FOR AN EVEN ORDER LINEAR DIFFERENTIAL EQUATION

GEORGE W. JOHNSON

Abstract. For an even order, two term equation \( L_n y = p(x) y \), \( p(x) > 0 \), \( x \) in \([0, \infty)\), the \( k \)th conjugate point function \( \eta_k(a) \) is defined and is shown to be a strictly increasing continuous function with domain \([0, b)\) or \([0, \infty)\). Extremal solutions are defined as nontrivial solutions with \( n-1+k \) zeros on \([a, \eta_k(a)]\), and are shown to have exactly \( n-1+k \) zeros, with even order zeros at \( a \) and \( \eta_k(a) \) and exactly \( k-1 \) odd order zeros in \((a, \eta_k(a))\), thus establishing that \( \eta_k(a) < \eta_{k+1}(a) \).

The differential equation considered in this paper is defined as follows:

Let \( p_1, \cdots, p_{n+1} \) be positive continuous functions defined on \([0, \infty)\), and let \( A_0 \) denote the set of all continuous functions defined on \([0, \infty)\). For \( y \) in \( A_0 \), define

\[
L_0 y = p_1 y.
\]

Assume that \( A_i \) and \( L_i y \) have been defined for \( i \leq k-1 \), and let \( A_k \) denote the set of all functions \( y \) for which \( L_{k-1} y \) has a continuous derivative on \([0, \infty)\). For \( y \) in \( A_k \), define

\[
L_k y = p_{k+1} (L_{k-1} y)' .
\]

The differential equation with which we are concerned is

\[
L_n y = p y,
\]

where \( p \) is a positive continuous function defined on \([0, \infty)\), and \( n \geq 4 \) is an even integer (cf. [3], [5], [6]).

From (1.2) and (1.3) it follows that \( L_n \) has the factored form

\[
L_n y = p_{n+1} (p_n \cdots p_3 (p_2 y)' \cdots y')'.
\]

For \( y \) in \( A_n \), the function \( L_i y \) is said to have a zero of multiplicity \( k \) at \( x = a \) if

\[
L_i y(a) = \cdots = L_{k-1+i} y(a) = 0 \quad \text{and} \quad L_{k+i} y(a) \neq 0 .
\]
Since the functions $p_1, \cdots, p_{n+1}$ are positive, it follows that if $L_{k-1}y(a) = L_{k-1}y(b) = 0$ for some $a < b$, then there is a number $c$ in $(a, b)$ for which $L_ky(c) = 0$. Moreover, if $L_ky$ has a zero of multiplicity $k$ at $x = a$, then $L_ky$ changes sign at $x = a$ if and only if $k$ is odd.

Denote by $N(k, a)$ the set of all points $x > a$ for which there is a non-trivial solution of (1.3), with zeros at $a$ and $x$, having at least $n - 1 + k$ zeros in $[a, x]$. The $k$th conjugate point of $a$, denoted by $\eta_k(a)$, is defined to be the infimum of the set $N(k, a)$; if $N(k, a)$ is empty, then $\eta_k(a) = +\infty$. Leighton and Nehari [4] studied the functions $\eta_k$ extensively for the equation $(ry'')'' - py = 0$ which is a special case of (1.3) with $n = 4, p_3 = r$, and $p_i = 1$ if $i \neq 3$.

In this paper we establish the existence of $\eta_k(a)$ whenever $N(k, a)$ is not empty, and we consider the properties of $\eta_k$ as a function of $a$. We also investigate the properties of the solutions which are extremal for $\eta_k(a)$, in the following sense. A solution $y$ of (1.3) will be called extremal if $y$ has a zero at $a$, a zero at $\eta_k(a)$ and $n - 1 + k$ zeros in $[a, \eta_k(a)]$.

We can now state the main results.

**Theorem 1.** If $y$ is an extremal solution for $\eta_k(a)$, then $y$ has even order zeros at $a$ and at $\eta_k(a)$; $y$ has exactly $n - 1 + k$ zeros in $[a, \eta_k(a)]$, with exactly $k - 1$ odd order zeros in $(a, \eta_k(a)); y$ is never zero in $(0, a)$ or in $(\eta_k(a), \infty)$.

**Theorem 2.** As a function of $a$, $\eta_k$ is a strictly increasing continuous function whose domain is of the form $[0, b)$, or $[0, \infty)$.

In order to establish Theorem 1, we will require the following results. For notational purposes, if $y$ is a solution with $n - 1 + k$ zeros at $r$ points $x_1 < \cdots < x_r$ we will denote by $m(x_i)$ the multiplicity of the zero of $y$ at $x_i$. Define the number

$$M(y) = \sum_{i \in I} m(x_i) + \sum_{i \in J} [m(x_i) - 1]$$

where

$I = \{i : m(x_i) \text{ is even}\}$ and $J = \{i : m(x_i) \text{ is odd}\}$.

**Lemma 1.** If $N(m, a)$ is nonempty, then for each $k = 1, \cdots, m$, there exists a $k$th conjugate point $\eta_k(a)$ and a nontrivial solution $y_k$ having the following properties.

(i) $a < \eta_k(a) \leq \eta_{k+1}(a)$, for $k = 1, \cdots, m - 1$.

(ii) $y_k$ has at least $n - 1 + k$ zeros in $[a, \eta_k(a)]$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
(iii) No nontrivial solution of (1.3) having a zero at a has more than \(n-2+k\) zeros in \([a, \eta_k(a))\).

For the proof, we observe that if \(N(m, a)\) is nonempty then \(N(k, a)\) is nonempty for \(k \leq m\), and if \(x_0 \in N(m, a)\) then there exist \(x_1 \leq x_2 \leq \cdots \leq x_k \leq x_0\) such that \(x_t \in N(i, a)\). Thus \(\eta_k(a)\) exists for each \(k < m\) and \(\eta_k(a) \leq \eta_{k+1}(a)\), establishing part (i). Part (iii) is a direct consequence of the definition of \(\eta_k(a)\). If \(N(k, a)\) is finite, or if there exists \(\epsilon > 0\) such that the intersection of \(N(k, a)\) with \((\eta_k(a), \eta_k(a)+\epsilon)\) is empty, then (ii) is immediate. Otherwise we observe that there exists a sequence \(\{x_t\}\) converging monotonically to \(\eta_k(a)\) and a sequence of solutions \(\{y_t\}\) having zeros at \(a\) and \(x_t\) with at least \(n-1+k\) zeros in \([a, x_t]\). With no loss of generality, we may assume that \(y_i\) has zeros at \(a = t_{i_1} < \cdots < t_{i_r} = x_i\) and that there are integers \(m_1, \cdots, m_r\) with \(m_1 + \cdots + m_r \geq n-1+k\) such that \(m(t_{i_j}) = m_j\) for all \(j = 1, \cdots, r, i\). Normalizing each solution \(y_i\), we may apply standard compactness arguments to obtain a nontrivial solution \(y\) of (1.3) and a subsequence \(\{y_{t_i}\}\) of \(\{y_i\}\) such that \(L_qy_{t_i}\) converges uniformly to \(L_qy\), for each \(q = 0, \cdots, n-1\), on \([a, x_t]\). Since limit points of the zeros \(L_qy_{t_i}\) are zeros of \(L_qy\) we have that \(y\) must have at least \(n-1+k\) zeros in \([a, \eta_k(a)]\) with a zero at \(a\) and a zero at \(\eta_k(a)\).

The following lemma, stated without proof, is due to Levin [5].

**Lemma 2.** There does not exist a nontrivial solution of (1.3) satisfying the following boundary conditions at \(x_1 < \cdots < x_r\):

\[
L_iy(x_i) = 0, \quad i = 0, \cdots, m(x_i)-1, \quad j = 1, \cdots, r
\]

if \(m(x_i)\) and \(m(x_r)\) are odd and

\(m(x_i)\) is even for \(i = 2, \cdots, r-1\).

It is an immediate consequence of Lemma 2 that if \(y\) is a nontrivial solution of (1.3) with \(n-1+k\) zeros at the points \(x_1 < \cdots < x_r\), then \(M(y) \leq n\), the number of odd order zeros of \(y\) must exceed \(k-2\) and will equal \(k-1\) only if \(m(x_1)\) and \(m(x_r)\) are even.

**Lemma 3.** If \(y\) is a nontrivial solution of (1.3) having \(n-1+k\) zeros at \(x_1 < \cdots < x_r\), such that \(M(y) < n\) then there exists a nontrivial solution of (1.3) with \(n-1+k\) zeros on \([x_1, x_r]\).

There are four cases to the proof, depending on whether the zeros at \(x_1\) and \(x_r\) are of even or odd multiplicity. Each case is treated in a similar fashion, so we will demonstrate the case in which \(m(x_1)\) is even and \(m(x_r)\) is odd. Let \(y_1, \cdots, y_n\) be a fundamental set of solutions of (1.3). By Lemma
is nonsingular for all $\varepsilon > 0$ sufficiently small, where $q_i = m(x_i)$, and for $i = 2, \cdots, r - 1$, $q_i = m(x_i) - 1$ if $m(x_i)$ is even, $q_i = m(x_i) - 2$ if $m(x_i)$ is odd and $m(x_i) \geq 3$, and $q_r = n - M(y) - 2$. Then for each $\varepsilon > 0$ there is a nontrivial solution vector $c(\varepsilon) = (c_1(\varepsilon), \cdots, c_n(\varepsilon))$ of $Y(\varepsilon)c(\varepsilon) = Y(0)c$ where $c = (c_1, \cdots, c_n)$ and $y = \sum_{i=1}^{n} c_i y_i$ is the solution with $n - 1 + k$ zeros in $[x_1, x_r]$. Letting $y(x, \varepsilon) = \sum_{i=1}^{n} c_i(\varepsilon)y_i(x)$, we have that as $\varepsilon \to 0$, $L_j y(x, \varepsilon)$ converges uniformly to $L_j y^*(x)$ for $j = 0, \cdots, n - 1$ where $y^*$ is a nontrivial solution of (1.3). Since $y - y^*$ satisfies the boundary conditions of Lemma 2, it must be the case that $y(x) = y^*(x)$ for all $x$. Now $y(x, \varepsilon)$ has an even order zero at each of the points $x_2, \cdots, x_{r-1}$, and hence for $\varepsilon$ sufficiently small, $y(x, \varepsilon)$ must change sign near each odd order zero of $y$. A simple count establishes that $y(x, \varepsilon)$ must have $n - 1 + k$ zeros in $[x_1, x_r - \varepsilon]$.

**Corollary 3.1.** If $y$ is an extremal solution of (1.3) for $\eta_k(a)$, then the zeros at $a$ and $\eta_k(a)$ are of even multiplicity.

If either $a$ or $\eta_k(a)$ is of odd multiplicity, then $M(y) < n$, which contradicts the fact that $y$ is extremal.

**Corollary 3.2.** If $y$ is an extremal solution for $\eta_k(a)$, then $y$ has exactly $n - 1 + k$ zeros in $[a, \eta_k(a)]$.

If $y$ has $n - 1 + r$ zeros in $[a, \eta_k(a)]$, and if $r > k$, then $m(\eta_k(a)) > r - k$. Using the techniques of Lemma 3 with the multiplicity $m(\eta_k(a)) - 1$ at $\eta_k(a)$ yields a solution with $n - 2 + r$ zeros in $[a, \eta_k(a) - \varepsilon]$. 
If an extremal solution for $\eta_k(a)$ has $m$ odd order zeros, then clearly $M(y) + m = n - 1 + k$, so that $m = k - 1$. If $y$ has a zero in either $(0, a)$ or $(\eta_k(a), \infty)$, then $y$ satisfies the boundary conditions of Lemma 2, yielding a contradiction. This completes the proof of Theorem 1.

**Lemma 4.** If $\eta_k(b) < \infty$, then there exists $\delta > 0$ such that for $a$ in $(b - \delta, b + \delta)$, $\eta_k(a) < \infty$.

Let $y$ be an extremal solution for $\eta_k(b)$ with zeros at $b = x_1 < \cdots < x_r = \eta_k(b)$. Then, for each $\epsilon$, sufficiently small, we define $y(x, \epsilon)$ to be a solution of (1.3) having zeros of multiplicity $m(x_i) - 1$ at $x_i$ if $i = 1$, $r$ or if $m(x_i)$ is odd, a zero of multiplicity $m(x_i)$ at $x_i$ if $m(x_i)$ is even, $1 < i < r$, and zero at $b + \epsilon$. Then we may write

$$y(x, \epsilon) = \sum_{i=1}^{n} c_i(\epsilon)y_i(x)$$

where \( \{y_1, \cdots, y_n\} \) is a fundamental set of solutions of (1.3), and with no loss of generality,

$$\sum_{i=1}^{n} c_i(\epsilon)^2 = 1.$$ 

Then as $\epsilon \to 0$, $L_i y(x, \epsilon)$ converges uniformly to $L_i y^*(x)$, where $y^*$ is a nontrivial solution of (1.3) satisfying

$$y^*(x) = \sum_{i=1}^{n} c_i y_i(x), \quad \sum_{i=1}^{n} c_i^2 = 1.$$ 

If $y^*(x) \neq ky(x)$ for all $x$, then there is a nontrivial linear combination of $y^*$ and $y$ satisfying the boundary conditions in Lemma 2, which is a contradiction. Then there is a $\delta > 0$ such that if $|\epsilon| < \delta$, $y(x, \epsilon)$ changes sign near each odd order zero of $y^*$ and near $\eta_k(b)$ since $y(x, \epsilon)$ has an odd order zero at $\eta_k(b)$ and $y^*$ has an even order zero there. A simple count yields that $y(x, \epsilon)$ has $n - 1 + k$ zeros in $[b + \epsilon, \eta_k(b) + \delta]$, and this completes the proof, since a similar argument holds for $b - \epsilon$.

If we define $y(x, \epsilon)$ as in Lemma 4 at the points $x_1, \cdots, x_r$ and require that $y(\eta_k(b) - \epsilon, \epsilon) = 0$ rather than $y(b - \epsilon) = 0$, for $\epsilon > 0$ then we obtain the following.

**Lemma 5.** If $\eta_k(b)$ exists, then there is a sequence $\{a_i\}$ converging to $b$ in $(b - \delta, b)$ such that $\eta_k(a_i) < \eta_k(b)$ for each $i$.

From the preceding discussion, we have for each $\epsilon > 0$, sufficiently small, that there exists a nontrivial solution $y(x, \epsilon)$ of (1.3) with $n - 1 + k$ zeros in $[b - \epsilon, \eta_k(b)]$. From Theorem 1, $y(x, \epsilon)$ is not an extremal solution for $\eta_k(b - \epsilon)$, hence $\eta_k(b - \epsilon) < \eta_k(b)$.
Corollary 5.1. If \( \eta_k(x) < \infty \) for all \( x \in [a, b] \) then \( \eta_k(a) < \eta_k(b) \).

Suppose to the contrary that \( \eta_k(a) \geq \eta_k(b) \). For some \( \varepsilon > 0 \), \( \eta_k(b-\varepsilon) < \eta_k(b) \), \( b-\varepsilon > a \) and \( S = \{ x > a : \eta_k(x) < \eta_k(b-\varepsilon) \} \) is nonempty, and hence if \( d = \inf S \), then \( \eta_k(d) = \eta_k(b-\varepsilon) \). If \( a < d \), then \( \eta_k(x) \geq \eta_k(b-\varepsilon) \geq \eta_k(d) \) for all \( x \) in \( (a, d) \), contradicting Lemma 6. Thus \( a = d \). Let \( x_i \) be a sequence in \( S \) converging monotonically to \( a \). Then arguments of Lemma 1 yield a sequence of extremal solutions \( y_i \) converging to a solution \( y \) of \((1.3)\) having \( n-1+k \) zeros in \([a, \eta_k(b-\varepsilon)]\). But this contradicts the definition of \( \eta_k(a) \) since \( \eta_k(a) > \eta_k(b) > \eta_k(b-\varepsilon) \).

Corollary 5.2. If \( \eta_k(b) < \infty \), then \( \eta_k(x) < \infty \) for all \( x \leq b \).

If, to the contrary, there exists an \( x < b \) for which \( N(x, k) \) is empty, let \( a = \sup \{ x < b : N(x, k) \text{ is empty} \} \). Then \( a < b \), and \( \eta_k(x) < \infty \) for all \( x \) in \( (a, b) \). From Corollary 6.1, there is a sequence \( \{ a_i \} \) in \( (a, b) \) such that \( a_{i+1} < a_i, \eta_k(a_{i+1}) < \eta_k(a_i) \) and \( a_i \) converges to \( a \) as \( i \) tends to \( \infty \). Then the techniques of Lemma 1 yield a solution \( y \) of \((1.3)\) such that \( y(a) = 0 \) and \( y \) has \( n-1+k \) zeros on \([a, \eta_k(b)]\), contradicting Lemma 4.

Corollary 5.3. \( \eta_k \) is continuous.

If as \( x \to a^-0 \), \( \eta_k(x) \to L < \eta_k(a) \), then the arguments of Lemma 1 yield a nontrivial solution of \((1.3)\) with \( n-1+k \) zeros in \([a, \eta_k(a)]\) since \( \eta_k \) is increasing. If as \( x \to a^+0 \), \( \eta_k(x) \to L > \eta_k(a) \), let \( \delta = \frac{1}{2}(L - \eta_k(a)) \). By Lemma 4, there exists \( \varepsilon > 0 \) and a solution of \( y(x, \varepsilon) \) having \( n-1+k \) zeros in \([a+\varepsilon, \eta_k(a)+\delta]\). This contradicts the fact that \( \eta_k \) is increasing.

Corollaries 5.1, 5.2, and 5.3 complete the proof of Theorem 2.

References

Department of Mathematics, University of South Carolina, Columbia, South Carolina 29208