\Lambda(p) SETS AND THE EXACT MAJORANT PROPERTY

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Abstract. Let \( \Gamma \) be a discrete abelian group. We prove that if \( 2 < p < \infty \) and \( E \subseteq \Gamma \), then \((E, p)\) has the exact majorant property if and only if \( E \) is a \( \Lambda(p) \) set.

Let \( G \) be a compact abelian group with dual group \( \Gamma \). If \( 1 \leq r < \infty \) and \( E \subseteq \Gamma \), let \( L_r^p(G) = \{ f : f \in L^p(G), f(y) = 0, \gamma \notin E \} \). If \( L_r^\infty(G) = L_r^2(G) \) for a set \( E \), we say \( E \) is a \( \Lambda(p) \) set. It follows that if \( L_r^\infty(G) = L_r^2(G) \) for some \( r, 1 \leq r < p \), then \( L_r^p(G) = L_r^\infty(G) = L_r^2(G) \) (cf. \[6, 37.7\]). If \( f, g \in L^p(G) \) and \( \hat{g} = |f| \) we say as in \[5\] that \( g \) is the exact majorant of \( f \). We say \((E, p)\) has the exact majorant property if whenever \( f \in L_r^p(G) \) then \( |f| \in (L_r^p(G))^* \), where \((L_r^p(G))^* = \{ \hat{f} : f \in L_r^p(G) \}\).

If \( p = 2 \) and \( E \subseteq \Gamma \), then \((E, 2)\) always has the exact majorant property since \( L_2^2(G) \) does. The following theorem and remark show that if \( 2 < p \leq \infty \) then \((E, p)\) has the exact majorant property only for special sets \( E \).

Theorem. Suppose \( 2 < p < \infty \); then \((E, p)\) has the exact majorant property if and only if \( E \) is a \( \Lambda(p) \) set.

Proof. Suppose \( E \) is a \( \Lambda(p) \) set. If \( f \in L_r^p(G) \) then \( f \in L_r^\infty(G) \), since \( L_r^\infty(G) \subseteq L_r^p(G) \) for \( p > 2 \). So \( \hat{f} \in (L_r^\infty(G))^* \) and \( |f| \in (L_r^p(G))^* \). But \( E \) is a \( \Lambda(p) \) set so \( L_r^\infty(G) = L_r^2(G) \) and \(|f| \in (L_r^p(G))^* = (L_r^2(G))^* \).

Conversely suppose \((E, p)\) has the exact majorant property. We wish to show \( L_r^\infty(G) = L_r^2(G) \). But \( L_r^\infty(G) \subseteq L_r^2(G) \), so it is sufficient to show \( L_r^2(G) \subseteq L_r^\infty(G) \).

Suppose \( f \in L_r^\infty(G) \). Define \( f_1 \) and \( f_2 \) in \( L_r^2(G) \) as follows:
\[
\hat{f}_1(\gamma) = \frac{1}{2}(\text{Re}\hat{f}(\gamma) + \text{Re}\hat{f}(\gamma)), \quad \hat{f}_2(\gamma) = \frac{1}{2}(\text{Re}\hat{f}(\gamma) - \text{Re}\hat{f}(\gamma)).
\]
Define \( f_3 \) and \( f_4 \) similarly with \( \text{Re}\hat{f} \) replaced by \( \text{Im}\hat{f} \). Then clearly \( f_1 \in L_r^2(G) \) for \( 1 \leq r \leq 4 \) and
\[
f = f_1 - f_2 + i(f_3 - f_4).
\]
Consider \( f_1 \), by \[3, 14.3.2\] or \[6, 36.5\], there exists a choice of numbers \( c_x \) of absolute value \( 1 \) such that if
\[
h_1(x) \sim \sum c_x f_1(\gamma)(x, \gamma).
\]

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then $h_1 \in L_E^p(G)$. But $(E, p)$ has the exact majorant property, so $|h_1| \in L_E^p(G)$.

Similarly $f_j \in L_E^p(G)$ for $2 \leq j \leq 4$. By equation (1), $f \in L_E^p(G)$.

**Remarks.** This theorem was proved for the special case when $p$ is an even integer $>2$ by Bachelis [1]. The proof actually gives the following: If $p$ is an even integer $>2$, then $E$ is a $\Lambda(p)$ set if and only if given $f \in L_E^p(G)$ there exists $g \in L_E^p(G)$ with $|f| \leq |g|$. (See [1, Theorem 3 and Lemma 1], or [2, Theorem 3].) The question of whether or not this characterization is also valid for $p$ not an even integer is open.

It is easy to show that $(E, \infty)$ has the exact majorant property if and only if $E$ is a Sidon set. It is known that a Sidon set is a $\Lambda(p)$ set for all $p < \infty$ [6, 37.10]. A natural question to ask is, if $(E, p)$ has the exact majorant property for all $p < \infty$, does $(E, \infty)$ have the exact majorant property? This question is answered in the negative since every infinite discrete abelian group contains a set $E$ which is $\Lambda(p)$ for all $p < \infty$ but not Sidon [4].

**Bibliography**


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